HARMONIC FUNCTIONS

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Abstract. In these notes, we explore the fundamental properties of harmonic functions, by using relatively elementary methods.

Contents

1. Introduction

Newton's *law of universal gravitation*, first published in his *Principia* in 1687, asserts that the force exerted on a point mass Q at $x \in \mathbb{R}^3$ by the system of finitely many point masses q_i at $y_i \in \mathbb{R}^3$, $(i = 1, \ldots, m)$, is equal to

$$F = \sum_{i=1}^{m} \frac{Cq_iQ}{|x - y_i|^2} \frac{x - y_i}{|x - y_i|}, \tag{1}$$

with a constant $C < 0$ (like masses attract). Here Q and q_i are understood as real numbers that measure how much mass the corresponding points have. The same law of interaction between point charges was discovered experimentally by Charles Augustin de Coulomb and announced in 1785, now with $C > 0$ (like charges repel). Note that the numerical value of the constant C depends on the unit system one is using to measure force, mass (or charge), and distance. After the introduction of the function

$$u(x) = \sum_{i=1}^{m} \frac{Cq_i}{|x - y_i|}, \tag{2}$$

into the theory of gravitation by Daniel Bernoulli in 1748, Joseph-Louis Lagrange noticed in 1773 that

$$F = -Q\nabla u, \qquad \text{at points} \quad x \neq y_i. \tag{3}$$

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Hence the function u, named later by Green the *potential function* and by Gauss the *potential*, completely describes the gravitational (or electrostatic) field. For a continuous distribution of charges with density ρ, vanishing outside some bounded set $\Omega \subset \mathbb{R}^3$, the potential becomes

$$u(x) = C \int_\Omega \frac{\rho(y)\,\mathrm{d}y}{|x-y|}. \tag{4}$$

As observed by Pierre-Simon Laplace in 1782, this potential satisfies

$$\Delta u = 0, \qquad \text{outside of } \Omega, \tag{5}$$

where $\Delta = \nabla^2 = \partial_1^2 + \ldots + \partial_n^2$ is arguably the most important differential operator in mathematics, called the *Laplace operator*, or the *Laplacian*. This equation came to be known as the *Laplace equation*, and its solutions are called *harmonic functions*. It should however be noted that the same equation had been considered by Lagrange in 1760 in connection with his study of fluid flow problems.

Exercise 1. Prove the statements in the observations of Lagrange and Laplace.

Laplace's result was completed by his student Siméon Denis Poisson in 1813, when Poisson showed that

$$\Delta u = -4\pi C\rho, \qquad \text{in } \mathbb{R}^3, \tag{6}$$

for smooth enough densities ρ. The idea[1] was to split the integral over Ω in (4) into two parts, one over the small ball $B_\varepsilon(x) = \{y \in \mathbb{R}^3 : |x-y| < \varepsilon\}$, and one over the complement $\Omega \setminus B_\varepsilon(x)$. Then the integral over the complement is harmonic by Laplace's result, and the other integral can be formally manipulated as

$$\Delta_x u(x) = C \int_{B_\varepsilon(x)} (\rho(y) - \rho(x)) \Delta_x \frac{1}{|x-y|}\,\mathrm{d}y + C\rho(x) \int_{B_\varepsilon(x)} \Delta_x \frac{1}{|x-y|}\,\mathrm{d}y, \tag{7}$$

where Δ_x means that the Laplacian acts on the x variable. Note that we are treating $1/|x-y|$ as if it was a smooth function. For the second integral of (7), a formal application of the divergence theorem gives

$$\int_{B_\varepsilon(x)} \Delta_x \frac{1}{|x-y|}\,\mathrm{d}y = \int_{\partial B_\varepsilon(x)} \partial_\nu \frac{1}{|x-y|}\,\mathrm{d}^2 y = 4\pi\varepsilon^2 (\frac{\partial}{\partial r}\frac{1}{r})|_{r=\varepsilon} = -4\pi, \tag{8}$$

where $\partial B_\varepsilon(x)$ is the boundary of the ball $B_\varepsilon(x)$, ∂_ν is the normal derivative at y with respect to the variable x, and $\mathrm{d}^2 y$ denotes the surface area element of $\partial B_\varepsilon(x)$. As for the first integral of (7), the function $1/|x-y|$ is harmonic except at $x = y$, and if $\rho(y) - \rho(x)$ vanishes sufficiently fast as $x \to y$, we can expect that the integral would come out as zero. The whole argument can be made rigorous, hence (6) is valid, e.g., provided that ρ is Hölder (or just Dini) continuous. Poisson argued that (6) is valid for any continuous function ρ, which is wrong, since there exist examples of continuous ρ for which (6) is *not* true. We will discuss this in the next set of notes.

Taking a slightly different viewpoint, if we started with the equation (6) for the unknown function u, with $f := -4\pi C\rho$, say, Hölder continuous, then the formula (4) gives a particular solution. This leads to the idea of a fundamental solution of a differential operator, that can be generalized to general dimensions and other linear operators.

In this and the following sets of notes, we will be concerned with *harmonic functions*, which are by definition the solutions of the Laplace equation

$$\Delta u = 0, \tag{9}$$

[1] Historians seem to regard that the argument was due to Laplace, who then "lent" it to his student.

in some domain Ω. We will also be interested in the inhomogeneous version of the Laplace equation, called the *Poisson equation*

$$\Delta u = f, \tag{10}$$

in some domain Ω. Note that the Poisson equation includes the Laplace equation as a special case, and the difference between two solutions (with the same f) of the Poisson equation is harmonic. There are tons of harmonic functions, meaning that the solutions of the Poisson equation are far from unique. In order to get uniqueness, i.e., as a convenient way of parameterizing the solution space of the Poisson equation, one introduces *boundary conditions*, which are conditions on the behaviour of u at the boundary $\partial\Omega$ of the domain. The common boundary conditions include

$$au + b\,\partial_\nu u = g, \qquad \text{on} \quad \partial\Omega, \tag{11}$$

with various choices of the functions a and b on the boundary. The case $a \equiv 1$ and $b \equiv 0$ is called the *Dirichlet*, $a \equiv 0$ and $b \equiv 1$ the *Neumann*, $a \equiv 1$ and $b > 0$ the *Robin*, and $a \equiv 1$ and $b < 0$ the *Steklov* boundary conditions.

2. Green's identities

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded open set with C^1 boundary $\partial\Omega$, and let $F : \overline{\Omega} \to \mathbb{R}^n$ be a vector field that is continuously differentiable in Ω and continuous up to the boundary, i.e., $F \in C^1(\Omega) \cap C(\overline{\Omega})$. Then the *divergence theorem* asserts

$$\int_\Omega \nabla \cdot F = \int_{\partial\Omega} F \cdot \nu, \tag{12}$$

where ν is the outward pointing unit normal to the boundary $\partial\Omega$. This can be thought of as an extension of the fundamental theorem of calculus to multidimensions. We remark that the regularity condition on $\partial\Omega$ can be considerably weakened, e.g., to include surfaces that consist of finitely many C^1 pieces. We will not discuss those issues here, as they are not necessary for our purposes. The same holds for the regularity conditions on F. The divergence theorem first appeared in Lagrange's 1860 work, and was proved in a special case by Gauss in 1813. The general 3-dimensional case was treated by Mikhail Vasilievich Ostrogradsky in 1826.

In a preliminary section of his groundbreaking 1828 *Essay*, George Green proved several reductions of 3-dimensional volume integrals to surface integrals, similar in spirit to the divergence theorem, and independently of Ostrogradsky. Nowadays, those are called *Green's identities* and best viewed as consequences of the divergence theorem. As a warmup, let $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and apply the divergence theorem to $F = \nabla\varphi$. Then we have $\nabla \cdot F = \Delta\varphi$ and $\nu \cdot F = \partial_\nu \varphi$, the latter denoting the (outward) normal derivative of φ, implying what can be called *Green's zeroth identity*[2]

$$\int_\Omega \Delta\varphi = \int_{\partial\Omega} \partial_\nu \varphi. \tag{13}$$

Similarly, letting $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and applying the divergence theorem to $F = u\nabla\varphi$, we get *Green's first identity*

$$\int_\Omega \nabla u \cdot \nabla\varphi + u\Delta\varphi = \int_{\partial\Omega} u\partial_\nu \varphi. \tag{14}$$

Interchanging the roles of u and φ in this identity, and subtracting the resulting identity from (14), we infer *Green's second identity*

$$\int_\Omega u\Delta\varphi - \varphi\Delta u = \int_{\partial\Omega} u\partial_\nu\varphi - \varphi\partial_\nu u, \qquad u, \varphi \in C^2(\Omega) \cap C^1(\overline{\Omega}). \tag{15}$$

[2] This name is not standard.

Note that (13) follows from (14) by putting $u \equiv 1$. The identities (14) and (15) can be considered as instances of, and are often called, *integration by parts* in n-dimensions.

Let us look at some simple consequences of Green's identities. First, the identity (13) gives a necessary condition for existence of a solution of the Neumann problem. For example, thinking of the Laplace equation, any solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of $\Delta u = 0$ must satisfy

$$\int_{\partial\Omega} \partial_\nu u = 0, \tag{16}$$

hence if it were to satisfy the Neumann boundary condition $\partial_\nu u = g$, the Neumann datum g should have the constraint that g has mean zero on the boundary $\partial\Omega$. Similarly, one can deduce that if the Poisson equation $\Delta u = f$ has a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with the homogeneous Neumann boundary condition $\partial_\nu u = 0$, then f necessarily has mean zero in the domain Ω.

Second, the identity (14) implies several *uniqueness theorems* for the Poisson equation. Recall that by linearity, the issue of uniqueness for the Poisson equation reduces to the study of the Laplace equation with homogeneous boundary conditions. By putting $u \equiv \varphi$ and $\Delta u = 0$ in (14), we have

$$\int_\Omega |\nabla u|^2 = \int_{\partial\Omega} u \partial_\nu u, \tag{17}$$

for $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with $\Delta u = 0$ in Ω. For example, if $u = 0$ on $\partial\Omega$, then the left hand side is zero, implying that $u = \text{const}$ in Ω. Using the condition $u = 0$ on the boundary once again makes $u \equiv 0$ in Ω, hence we get uniqueness of solutions to the Dirichlet problem for the Poisson equation. On the other hand, if $\partial_\nu u = 0$ on $\partial\Omega$, then by the same argument we have $u = \text{const}$, implying that any two solutions (with suitable regularity) of the Neumann problem for the Poisson equation differ by a constant. This cannot be strengthened, since one can explicitly check that if u is a solution of $\Delta u = 0$ with $\partial_\nu u = 0$ on the boundary, then so is $u + \alpha$ for any constant α.

The aforementioned uniqueness results show that the *Cauchy problem* for $\Delta u = 0$, where we specify both u and $\partial_\nu u$ at the boundary of the domain, is in general *not* solvable. It would be analogous to overdetermined equations, where one has more equations than unknowns.

Exercise 2. Prove a uniqueness theorem for the Robin problem for the Poisson equation. What if one specifies a Dirichlet condition on one part of the boundary, and a Neumann condition on the rest?

3. Fundamental solutions

We saw in the introduction the plausibility that the function

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)\,\mathrm{d}y}{|x-y|}, \tag{18}$$

satisfies the equation $\Delta u = f$, cf. (4) and (6). In other words, at least formally speaking, the operation sending f to u in (18) inverts the action of Δ. We want to start now a systematic and rigorous study of this phenomenon.

Definition 1. A *fundamental solution* (or *elementary solution*) of the Laplacian in n dimensions is a locally integrable function $E \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} E(y) \Delta\varphi(y)\,\mathrm{d}y = \varphi(0), \tag{19}$$

for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$, where $\mathscr{D}(\mathbb{R}^n)$ is the space of compactly supported smooth functions on $\mathbb{R}^n$.

Let us make some simple observations around this definition. From our previous discussion (18), we know that $E(x) = -\frac{1}{4\pi|x|}$ is a good candidate for a fundamental solution in 3 dimensions. Moreover, replacing $\varphi(y)$ by $\varphi(x+y)$ in the integral in (19), we see that if E is a fundamental solution, then we have

$$\int_{\mathbb{R}^n} E(y-x)\Delta\varphi(y)\,\mathrm{d}y = \varphi(x), \tag{20}$$

a formula that extends (18) in a certain sense. Now suppose that E is twice continuously differentiable in a neighbourhood of some point x. Then by Green's second identity (15) we have

$$\varphi(0) = \int_{\mathbb{R}^n} E(y)\Delta\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \Delta E(y)\varphi(y)\,\mathrm{d}y, \tag{21}$$

for all smooth φ that are supported in a small neighbourhood of x. If $x \neq 0$, then by considering a possibly smaller neighbourhood of x, we can ensure that $\varphi(0) = 0$. This implies that $\Delta E = 0$ in a neighbourhood of x. Hence, if a fundamental solution E is smooth on an open set ω that does not contain the origin, then E must be harmonic on ω.

There is an interpretation of fundamental solutions that can also be used to motivate the whole concept. We will only give a heuristic reasoning, although everything can be made rigorous by generalizing the notion of derivatives to work with "rough" objects such as measures and distributions. In 3-dimensions, by (2), our candidate fundamental solution $E(x) = -\frac{1}{4\pi|x|}$ is in fact the potential produced by a point charge of quantity $q = -1/(4\pi C)$ located at the origin. From Poisson's investigations (6), we expect that the potential u generated by the charge distribution ρ satisfies $\Delta u = -4\pi C\rho = \rho/q$. Then a question is: What is the charge density ρ corresponding to the charge q concentrated at the origin? Certainly, $\rho(x) = 0$ for $x \neq 0$, and we must have $\int \rho = q$. Such a function does not exist, since the condition $\rho(x) = 0$ for $x \neq 0$ already implies $\int \rho = 0$, whatever the value $\rho(0)$ is. Nevertheless, if such a thing existed, we expect to have $\Delta E = \rho/q$. Following Paul Dirac, we introduce the notation $\delta(x) = \rho(x)/q$. It has many names, including Dirac's delta function, the Dirac measure, and delta distribution. Note that δ is simply a normalized version of ρ, so that $\int \delta = 1$. To get further insight, for $\varphi \in C(\mathbb{R}^n)$, we formally manipulate

$$\int_{\mathbb{R}^n} \varphi(x)\delta(x)\,\mathrm{d}x = \int_{B_\varepsilon} \varphi(x)\delta(x)\,\mathrm{d}x = \int_{B_\varepsilon} \delta(x)[\varphi(0) + e(x)]\,\mathrm{d}x, \tag{22}$$

where $B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$ and $\sup_{x\in B_\varepsilon} |e(x)| \to 0$ as $\varepsilon \to 0$. Continuing formally, the latter property implies

$$\Big|\int_{B_\varepsilon} \delta(x)e(x)\,\mathrm{d}x\Big| \leq \sup_{x\in B_\varepsilon} |e(x)| \int_{B_\varepsilon} \delta(x)\,\mathrm{d}x = \sup_{x\in B_\varepsilon} |e(x)| \to 0 \qquad \text{as} \quad \varepsilon \to 0, \tag{23}$$

and hence

$$\int_{\mathbb{R}^n} \varphi(x)\delta(x)\,\mathrm{d}x = \varphi(0)\int_{B_\varepsilon} \delta(x)\,\mathrm{d}x + \int_{B_\varepsilon} \delta(x)e(x)\,\mathrm{d}x = \varphi(0). \tag{24}$$

In light of this, we can interpret (21) as saying that $\Delta E = \delta$, under the formal assumption that (21) is valid in general (i.e., even for nonsmooth E). In fact, one can view the definition (19) of fundamental solutions as a way to give a precise sense to the formal expression $\Delta E = \delta$.

We look for a spherically symmetric fundamental solution of the Laplace operator.

Exercise 3. Let $\phi : (a,b) \to \mathbb{R}$ be a twice differentiable function with $a \geq 0$, and define $u(x) = \phi(|x|)$ for $x \in \mathbb{R}^n$ with $a < |x| < b$, where $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$. Show that

$$\Delta u(x) = \phi''(|x|) + \frac{n-1}{|x|}\phi'(|x|), \qquad \text{for} \qquad a < |x| < b.$$

Find all solutions of $\Delta u = 0$, where u is of the above form with $(a,b) = (0,\infty)$. ⊘

This exercise shows that the function $E(x) = \phi(|x|)$ with

$$\phi(r) = \begin{cases} C_n r^{2-n} & \text{if } n \neq 2, \\ C_2 \log r & \text{if } n = 2, \end{cases} \tag{25}$$

satisfies $\Delta E = 0$ in $\mathbb{R}^n \setminus \{0\}$. Now we need to check if the constants C_n can be tuned so that E is indeed a fundamental solution. Note that E is locally integrable, as the singularity at 0 is not strong enough. Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ be a compactly supported smooth function in $\mathbb{R}^n$. Choose $R > 0$ so large that $\operatorname{supp} \varphi \subset B_R$, and let $\varepsilon > 0$ be small. We apply Green's second identity (15) on $\Omega = B_R \setminus \overline{B_\varepsilon}$, with $u = E$ and $v = \varphi$, to get

$$\int_\Omega E\Delta\varphi - \varphi\Delta E = \int_{\partial B_R} (E\partial_\nu\varphi - \varphi\partial_\nu E) - \int_{\partial B_\varepsilon} (E\partial_r\varphi - \varphi\partial_r E), \tag{26}$$

where we have taken into account that the radial derivative ∂_r is the opposite of the outer normal derivative ∂_ν at the inner boundary of Ω. Since E is harmonic in $\mathbb{R}^n \setminus \{0\}$, the term with ΔE vanishes. The term with the integral over ∂B_R vanishes too, because $\operatorname{supp} \varphi \subset B_R$, resulting in

$$\int_\Omega E\Delta\varphi = \int_{\partial B_\varepsilon} \varphi\partial_r E - \int_{\partial B_\varepsilon} E\partial_r\varphi. \tag{27}$$

As $\varepsilon \to 0$, the left hand side tends to the integral of $E\Delta\varphi$ over $\mathbb{R}^n$ because E is locally integrable. For the last term, we have

$$\Big|\int_{\partial B_\varepsilon} E\partial_r\varphi\Big| = \varepsilon^{2-n}\Big|C_n \int_{\partial B_\varepsilon} \partial_r\varphi\Big| = \varepsilon^{2-n}\Big|C_n \int_{B_\varepsilon} \Delta\varphi\Big| \leq C\varepsilon^{2-n}\varepsilon^n \|\Delta\varphi\|_\infty \to 0, \tag{28}$$

where in the second step we used what we called Green's zeroth identity (13), and $C > 0$ is some constant. For the first term in the right hand side of (27), we have

$$\int_{\partial B_\varepsilon} \varphi\partial_r E = (2-n)C_n\varepsilon^{1-n} \int_{\partial B_\varepsilon} \varphi = (2-n)C_n\varepsilon^{1-n}|S^{n-1}|\varepsilon^{n-1}(\varphi(0) + o(1)), \tag{29}$$

which converges to $(2-n)C_n|S^{n-1}|\varphi(0)$, where $|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the $n-1$ dimensional surface area of the unit $n-1$ sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. So choosing $C_n = \frac{1}{(2-n)|S^{n-1}|}$ for $n \neq 2$, we can guarantee

$$\int_{\mathbb{R}^n} E\Delta\varphi = \varphi(0). \tag{30}$$

The final expression for the fundamental solution we obtained is

$$E(x) = \begin{cases} \frac{1}{(2-n)|S^{n-1}|}|x|^{2-n} & \text{if } n \geq 3, \\ \frac{1}{2\pi}\log|x| & \text{if } n = 2, \\ \frac{1}{2}|x| & \text{if } n = 1. \end{cases} \tag{31}$$

Exercise 4. Derive the value of C_2.

4. Green's formula

In this section, we look into the possibility of using fundamental solutions of the Laplace operator as one of the functions in Green's second identity (15). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary. Let $y \in \Omega$, and we put $\varphi(x) = E_y(x) := E(x-y)$ into Green's second identity, where E is the special fundamental solution (31). Since $E_y \notin C^2(\Omega)$, in order to apply the identity, we excise a small ball $B_\varepsilon(y)$ of radius $\varepsilon > 0$ centred at y from Ω, resulting in $\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon(y)}$. Assuming that $\varepsilon > 0$ is so small that $\overline{B_\varepsilon(y)} \subset \Omega$, and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, an application of Green's second identity on Ω_ε gives

$$\int_{\Omega_\varepsilon} u\Delta E_y - E_y \Delta u = \int_{\partial\Omega} (u\partial_\nu E_y - E_y \partial_\nu u) - \int_{\partial B_\varepsilon(y)} (u\partial_r E_y - E_y \partial_r u), \tag{32}$$

where we have taken into account that the boundary of Ω_ε consists of two parts $\partial\Omega$ and $\partial B_\varepsilon(y)$, and ∂_r is the radial derivative centred at y. We already know from the preceding section how to take the limit $\varepsilon \to 0$ in (32), but let us repeat the arguments here. Since E_y is harmonic except at $x = y$, the term with $u\Delta E_y$ vanishes. As for the other term in the left hand side, we have

$$\int_{B_\varepsilon(y)} |E_y \Delta u| \leq C_n \int_0^\varepsilon r^{2-n} r^{n-1} (\Delta u(y) + o(1))\, \mathrm{d}r = O(\varepsilon^2), \tag{33}$$

meaning that the integral

$$\int_\Omega E_y \Delta u = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} E_y \Delta u, \tag{34}$$

can be interpreted either as an absolutely convergent improper Riemann integral, or an ordinary Lebesgue integral. For the last terms in (32), by using the continuity of ∇u and u, and the definition of E_y, we infer

$$\int_{\partial B_\varepsilon(y)} |E_y \partial_r u| \leq C_n \varepsilon^{2-n} |S^{n-1}| \varepsilon^{n-1} (|\nabla u(y)| + o(1)) = O(\varepsilon) \tag{35}$$

and

$$\int_{\partial B_\varepsilon(y)} u\partial_r E_y = (u(y) + o(1)) C_n (2-n) \varepsilon^{1-n} |S^{n-1}| \varepsilon^{n-1} = u(y) + o(1), \tag{36}$$

resulting in what is called *Green's formula* or *Green's third identity*

$$u(y) = \int_\Omega E_y \Delta u + \int_{\partial\Omega} u\partial_\nu E_y - \int_{\partial\Omega} E_y \partial_\nu u, \tag{37}$$

for $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. We emphasize here that as stated, Green's formula is not tied to any differential equation, and what it delivers is a way to represent an arbitrary function as the sum of certain special kinds of functions, called *potentials*. We call the first term in the right hand side the *Newtonian potential*, the second term the *double layer potential*, and the last term the *single layer potential*. An immediate consequence of this formula is that if $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, then $u \in C^\infty(\Omega)$, because double- and single layer potentials are smooth in the interior of the domain. Results such as this one, that allow one to conclude higher regularity from a lower regularity and a differential equation, are called *regularity theorems*.

As a special case of (37), if we let $u \in C^2(\mathbb{R}^n)$ compactly supported, and take Ω to be a large ball containing the support of u, then we get

$$u(y) = \int_{\mathbb{R}^n} E_y \Delta u = \int_{\mathbb{R}^n} E(y - x) \Delta u(x)\, \mathrm{d}x, \tag{38}$$

which is of course simply a translated version of (30).

Exercise 5. One can extract a regularity result for harmonic functions from (38) as follows. Let $B_r(y)$ be a small ball centred at y, and let $\chi \in \mathscr{D}(B_{2r}(y))$ satisfy $\chi \equiv 1$ in $B_r(y)$ and $0 \leq \chi \leq 1$ everywhere. Here $\mathscr{D}(B_{2r}(y))$ is the space of smooth functions whose support is contained in $B_{2r}(y)$. Let v be C^2 and harmonic in $B_{2r}(y)$. Apply (38) to $u = \chi v$ and deduce that v is C^∞ in a neighbourhood of y.

Exercise 6. Let Ω and u be as in Green's formula (37), and in addition, let $\Delta u = 0$ in Ω. Then by using Green's formula, prove that $u \in C^\infty(\Omega)$ with

$$\sup_{y\in K} |\partial^\alpha u(y)| \leq C(\sup_\Omega |u| + \sup_\Omega |\nabla u|), \tag{39}$$

for all multi-indices α and compact sets $K \subset \Omega$, with the constant C possibly depending on α and K.

5. Mean value property

In this section, we will prove the mean value theorem of Gauss, and derive some of its direct consequences.

Theorem 2 (Gauss 1840). *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in C^2(\Omega)$ with $\Delta u = 0$ in Ω. Then for any ball $\overline{B_r(y)} \subset \Omega$, we have*

$$\int_{\partial B_r(y)} \partial_\nu u = 0, \tag{40}$$

and

$$u(y) = \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u = \frac{1}{|B_r|} \int_{B_r(y)} u. \tag{41}$$

Proof. The property (40) is immediate from (13). Then the first equality in (41) follows from Green's formula (37) and the radial symmetry of E, and we get the second equality by a radial integration. □

The property (41) is called the *mean value property* of harmonic functions. Both (40) and (41) can be compared to Cauchy's theorem (or perhaps Cauchy's integral formula) in complex analysis. Note that any of the equalities implies the other by either integration or differentiation. We will later prove that the mean value property characterizes harmonicity, which would be an analogue of Morera's theorem. For now, let us look at some of its immediate consequences.

Lemma 3 (Harnack inequality). *Let $u \in C^2(\Omega)$ be a nonnegative function harmonic in Ω. Let $\overline{B_R(y)} \subset \Omega$ with $R > 0$, and $x \in B_R(y)$. Then we have*

$$u(x) \leq \left(\frac{R}{R - |x-y|}\right)^n u(y) = \left(\frac{1}{1-k}\right)^n u(y), \tag{42}$$

where $k = |x-y|/R$.

Proof. From the mean value property and the positivity of u, with $r = R - |x-y|$ we have

$$u(x) = \frac{1}{|B_r|} \int_{B_r(x)} u \leq \frac{1}{|B_r|} \int_{B_R(y)} u = \frac{|B_R|}{|B_r|} u(y), \tag{43}$$

establishing the claim. □

Exercise 7. Let $u \in C^2(\Omega)$ be harmonic in a bounded domain Ω. By using the Harnack inequality show that unless u is constant, it cannot achieve its extremums in Ω.

For nonnegative *entire* harmonic functions, we can apply (42) with fixed x, y, and take the limit $R \to \infty$ to get the following result.

Corollary 4 (Liouville's theorem[3]). *Let $u \in C^2(\mathbb{R}^n)$ be nonnegative and harmonic in $\mathbb{R}^n$. Then u is constant.*

[3] Sometimes attributed to Picard

Exercise 8. Show that the conclusion of Liouville's theorem still holds if we relax the condition "nonnegative" to "bounded from above or below".

Exercise 9. Let $u \in C^2(\mathbb{R}^n)$ be harmonic in $\mathbb{R}^n$, satisfying $u(x) \geq -p(|x|)$ for some nondecreasing function $p : [0, \infty) \to [0, \infty)$. Show that $u(x) \leq 2^n(u(0) + p(2|x|))$ for $x \in \mathbb{R}^n$.

6. Maximum principles

Definition 5. A continuous function $u \in C(\Omega)$ is called *subharmonic* in Ω, if for any $y \in \Omega$, there exists $r^* > 0$ such that

$$u(y) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u, \qquad 0 < r < r^*. \tag{44}$$

Exercise 10. Let $u \in C^2(\Omega)$. Show that u is subharmonic in Ω if and only if $\Delta u \geq 0$ in Ω.

Theorem 6 (Strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in C(\Omega)$ be subharmonic in Ω. If $u(z) = \sup_\Omega u$ for some $z \in \Omega$, then u is constant in the connected component of Ω that contains z.*

Proof. Let $M = \sup_\Omega u$. Then by hypothesis the set $\Sigma = \{y \in \Omega : u(y) = M\}$ is nonempty and closed. Now suppose that $y \in \Sigma$. Then by subharmonicity of u we have

$$\frac{1}{|B_r|} \int_{B_r(y)} M = u(y) \leq \frac{1}{|B_r|} \int_{B_r(y)} u, \tag{45}$$

for small $r > 0$, giving

$$\int_{B_r(y)} (u(x) - M) \, \mathrm{d}x \geq 0. \tag{46}$$

This means that $u \equiv M$ in $B_r(y)$, hence Σ is open. □

If Ω is bounded and if u is continuous up to the boundary of Ω, then u has its maximum in $\overline{\Omega}$. Since a maximum in the interior of Ω would imply that u is constant in the connected component of Ω containing the maximum, in any case the maximum value is attained at the boundary of Ω.

Corollary 7 (Weak maximum principle)**.** *Let Ω be a bounded open set, and let $u \in C(\overline{\Omega})$ be subharmonic in Ω. Then*

$$\sup_\Omega u = \max_{\partial\Omega} u, \tag{47}$$

i.e., u achieves its maximum at the boundary.

An immediate consequence is a *uniqueness theorem* for the Dirichlet problem. Indeed, by linearity the question reduces to the uniqueness for the homogeneous problem $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$. Then we can apply the weak maximum principle to u and to $-u$ to infer $u = 0$. Note that we require $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and that Ω be a bounded open set, which is weaker than the conditions in the uniqueness proof using the identity (17). Maximum principles applied to $-u$ are sometimes called *minimum principles* for u.

If we apply maximum principles to the difference between two functions, we obtain so-called *comparison principles*, which are so useful that they deserve a statement of their own.

Corollary 8 (Comparison principle)**.** *Let Ω be a bounded open set, and let u and v be elements of $C^2(\Omega) \cap C(\overline{\Omega})$. Assume that $\Delta u \geq \Delta v$ in Ω and that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

As an application, we prove the following *a priori* bound.

Corollary 9. *Let Ω be a bounded open set. Then for $u \in C^2(\Omega) \cap C(\overline{\Omega})$ we have*

$$\sup_{\Omega} |u| \leq C \sup_{\Omega} |\Delta u| + \sup_{\partial\Omega} |u|, \tag{48}$$

where $C > 0$ is a constant depending only on Ω.

Proof. Suppose that Ω is contained in the strip $\{x \in \mathbb{R}^n : 0 < x_1 < d\}$ with some $d > 0$. Let $v(x) = \alpha - \gamma x_1^2$ with constants α and γ to be determined. We have $\Delta v = -2\gamma$, meaning that the choice $\gamma = \frac{1}{2} \sup_{\Omega} |\Delta u|$ would ensure that $\Delta u \geq \Delta v$ in Ω. Then in order to have $u \leq v$ on $\partial\Omega$, we put $\alpha = \sup_{\partial\Omega} |u| + \gamma d^2$, which gives the bound $u \leq v \leq \alpha$ in Ω. The same function v works also for $-u$. □

7. Green's function approach

Adding (15) to (37), we get the *generalized Green formula*

$$u(y) + \int_{\Omega} u \Delta\varphi = \int_{\Omega} \Phi_y \Delta u + \int_{\partial\Omega} u \partial_\nu \Phi_y - \int_{\partial\Omega} \Phi_y \partial_\nu u, \tag{49}$$

for $u, \varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with $\Phi_y(x) = E(x - y) + \varphi(x)$ and $y \in \Omega$. Recall that Ω is assumed to be a bounded C^1 domain in $\mathbb{R}^n$. Consider the *Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{50}$$

where $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ are given functions, respectively called the *source term* and *Dirichlet datum.* Then assuming that u satisfies (50), and applying (49) to it, we observe that

$$u(y) = \int_{\Omega} \Phi_y f + \int_{\partial\Omega} g \, \partial_\nu \Phi_y, \tag{51}$$

provided that $\Delta\varphi = 0$ in Ω and $\Phi_y = 0$ on $\partial\Omega$. The latter conditions are equivalent to

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega, \\ \varphi = -E_y & \text{on } \partial\Omega. \end{cases} \tag{52}$$

So this approach potentially reduces the general Dirichlet problem (50) to a set of special Dirichlet problems (52). Note that we have to solve one special problem for each $y \in \Omega$. For this reason, it is preferable to denote φ in (52) by φ_y, so in particular, $\Phi_y(x) = E_y(x) + \varphi_y(x)$. The function $(y, x) \mapsto \Phi_y(x)$ is called *Green's function* for the Dirichlet problem (50), and $(y, x) \mapsto \varphi_y(x)$ is called the corresponding *correction function.*

We remark that formally, the problem (52) is equivalent to

$$\begin{cases} \Delta\Phi_y = \Delta E_y \equiv \delta_y & \text{in } \Omega, \\ \Phi_y = 0 & \text{on } \partial\Omega, \end{cases} \tag{53}$$

where $\delta_y(x) = \delta(x - y)$ is the shifted delta distribution.

Now, in order to justify the whole thing, we need to address the following questions.

(i) Does Φ_y exists, i.e., is the problem (52) solvable?
(ii) Supposing that Φ_y exists, does the function u defined by (51) solve the problem (50)?

In general, the first question is essentially as difficult as solving the general problem (50), but when Ω is simple, e.g., a ball or a half space, we can solve (53) explicitly, and hence we will have an integral formula for the solution of (50). On the other hand, there is a quite persuasive physics reasoning on the solvability of (53):

> To convince ourselves that there does exist such a function as we have supposed Φ_y to be; conceive the surface to be a perfect conductor put in communication with the earth, and a unit of positive electricity to be concentrated in the point y, then the total potential function arising from y and from the electricity it will induce upon the surface, will be the required value of Φ_y (Green 1828).

We will see in the next set of notes that this intuition is correct if the boundary of Ω is nice in a certain sense. For highly nonsmooth boundaries though, the problem (52) is not always solvable.

As for Question (ii), it can be answered without much difficulty when the domain boundary is nice, but we will not go into details here as the next sections do not depend on it. The main use we have for Green's function is to give a derivation of Poisson's formula, whose validity will be verified independently. Moreover, solvability of (50) will be treated by more direct methods.

Remark 10. Let $G(y,x) = \Phi_y(x)$ be Green's function considered as a locally integrable function on $\Omega \times \Omega$. Then assuming that G exists, we have

- G is unique and $G < 0$.
- $G(y,x) > E(x-y)$ if $n \geq 3$ and $G(y,x) > E(x-y) - \frac{1}{2\pi}\log \operatorname{dist}(y,\partial\Omega)$ if $n = 2$.
- $G(x,y) = G(y,x)$ for $x \neq y$.

Uniqueness follows from the uniqueness theorem for (52). Since $\varphi|_{\partial\Omega} = -E_y|_{\partial\Omega} > 0$ for $n \geq 3$, the maximum principle says that $\varphi > 0$, which implies $G(y,x) > E(x-y)$. For $n = 2$, we have $E_y|_{\partial\Omega} < \frac{1}{2\pi}\log \operatorname{dist}(y,\partial\Omega)$, so $G(y,x) > E(x-y) - \frac{1}{2\pi}\log \operatorname{dist}(y,\partial\Omega)$. The negativity $G < 0$ is because the function $G_y(x) = G(y,x)$ is harmonic in Ω except at $x = y$, and satisfies $G_y|_{\partial\Omega} = 0$ and $G_y(x) \to -\infty$ as $x \to y$. For an elementary proof of the symmetry $G(x,y) = G(y,x)$ we refer to Evans (Chapter 2, Theorem 13) or Han (Lemma 4.1.5).

Example 11. Let us find Green's function for $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$, the "upper half space". The defining condition (53) for Green's function arose from discussions about bounded domains, but the equations (53) make perfect sense even in unbounded domains. Let $y \in \Omega$. Then we have $\Delta E_y = \delta_y$, but we have to correct it by a function φ harmonic in Ω so as to have $(E_y+\varphi)|_{x_n=0} = 0$. It is obvious that $\varphi(x) = -E_{-y}(x) = -E(x+y)$ works, since $|x-y| = |x+y|$ if $x_n = 0$, and E_{-y} is harmonic in Ω. Thus Green's function is $G_y(x) = E(x-y) - E(x+y)$. For $n \geq 3$ we have

$$\partial_n |x-y|^{2-n} = (2-n)|x-y|^{-n}(x_n - y_n), \tag{54}$$

and so

$$\partial_\nu G_y(x) = -\partial_n G_y(x) = \frac{2y_n}{|S^{n-1}| \cdot |x-y|^n}, \tag{55}$$

which is called the *Poisson kernel* for the half space. For $n = 2$ we have

$$\partial_2 \log|x-y| = |x-y|^{-2}(x_2 - y_2), \qquad \text{hence} \qquad \partial_\nu G_y(x) = \frac{2y_2}{2\pi |x-y|^2}, \tag{56}$$

meaning that (55) holds for all $n \geq 2$.

Green's function approach can be extended to the *Neumann problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \partial\Omega, \end{cases} \tag{57}$$

where $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ are given functions. Let us note that in order for a solution u to exist, the data must satisfy the consistency condition

$$\int_\Omega f = \int_{\partial\Omega} g, \tag{58}$$

that follows from (13), and that the solution u is determined by (57) only up to a constant. Imitating what we did for the Dirichlet case, for any solution u of (57), from (49) we get

$$u(y) = \int_\Omega \Phi_y f - \int_{\partial\Omega} \Phi_y g, \tag{59}$$

provided that $\Delta\varphi = 0$ in Ω and $\partial_\nu \Phi_y = 0$ on $\partial\Omega$, where $\Phi_y = E_y + \varphi$. The analogue of (52) is

$$\begin{cases} \Delta\varphi_y = 0 & \text{in } \Omega, \\ \partial_\nu \varphi_y = -\partial_\nu E_y & \text{on } \partial\Omega, \end{cases} \tag{60}$$

where $\Phi_y(x) = E_y(x) + \varphi_y(x)$. This problem is not solvable, since (13) requires

$$\int_{\partial\Omega} \partial_\nu \varphi_y = 0, \tag{61}$$

but (37) with $u \equiv 1$ implies

$$\int_{\partial\Omega} \partial_\nu E_y = 1. \tag{62}$$

Therefore in order to have a chance at solvability, we have to replace the problem (60) by

$$\begin{cases} \Delta\varphi_y = 0 & \text{in } \Omega, \\ \partial_\nu \varphi_y = \frac{1}{|\partial\Omega|} - \partial_\nu E_y & \text{on } \partial\Omega. \end{cases} \tag{63}$$

Assuming that such φ_y exists for each $y \in \Omega$, we put $N(y, x) = N_y(x) = E_y(x) + \varphi_y(x)$, which is called *Neumann's function* or *Green's function of the second kind*. It is easy to see that the analogue of (53) is

$$\begin{cases} \Delta N_y = \delta_y & \text{in } \Omega, \\ \partial_\nu N_y = \frac{1}{|\partial\Omega|} & \text{on } \partial\Omega, \end{cases} \tag{64}$$

and the analogue of (51) is

$$u(y) = \int_\Omega N_y f - \int_{\partial\Omega} N_y g + \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u. \tag{65}$$

The last term is natural, as it gives a possibility to specify the mean of u over the boundary of the domain.

Exercise 11. Devise an analogous "Robin's function" approach for the Robin problem.

8. Poisson's formula

By using Green's function approach, in this section we will solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B_r, \\ u = g & \text{on } \partial B_r, \end{cases} \tag{66}$$

in the ball $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, where $g \in C(\partial B_r)$ is a given function. It will be a stepping stone to solving the Dirichlet problem in general domains. Fix $y \in B_r$, and we look for the Green function in the form

$$G_y(x) = E(x - y) - qE(x - y^*) + c, \tag{67}$$

with $y^* = \lambda y$, where q, c and λ are real numbers possibly depending on y. If λ is so large that y^* is outside B_r, then the second term is harmonic in B_r, so all we need to do is to ensure that $G_y = 0$ on ∂B_r. We see that

$$|x - \lambda y|^2 = |x|^2 + \lambda^2|y|^2 - 2\lambda x \cdot y = r^2 + \lambda^2|y|^2 - 2\lambda x \cdot y, \tag{68}$$

is a constant multiple of $|x-y|^2 = r^2 + |y|^2 - 2x \cdot y$ for all $x \in \partial B_r$, if $\lambda = 1$ or $\lambda|y|^2 = r^2$. Since we want $\lambda > 1$, the latter is clearly our choice, with which we then have

$$|x - y^*| = \sqrt{\lambda}|x-y| = \frac{r}{|y|}|x-y|. \tag{69}$$

This implies

$$E(x-y^*) = \big(\frac{|y|}{r}\big)^{n-2} E(x-y), \tag{70}$$

for $n \geq 3$, and

$$E(x-y^*) = E(x-y) + \frac{1}{2\pi}\log\big(\frac{r}{|y|}\big), \tag{71}$$

for $n = 2$. Then from (67) it is easy to figure out the values of q and c that ensures $G_y = 0$ on ∂B_r, resulting in

$$G_y(x) = E(x-y) - \big(\frac{r}{|y|}\big)^{n-2} E(x-y^*) + \begin{cases} 0 & \text{for } n \geq 3, \\ \frac{1}{2\pi}\log\big(\frac{r}{|y|}\big) & \text{for } n = 2, \end{cases} \tag{72}$$

where $y^* = \frac{r^2}{|y|^2}y$. The formula (51) involves $\partial_\nu G_y$, so let us compute it. Assuming $n \geq 3$, for $a \in B_r$ and $x \in \partial B_r$, we have

$$\partial_\nu |x-a|^{2-n} = (2-n)|x-a|^{-n}\frac{|x|^2 - a\cdot x}{|x|} = \frac{(2-n)(r^2 - a \cdot x)}{r|x-a|^n}, \tag{73}$$

and hence

$$\partial_\nu E(x-y) = \frac{r^2 - y\cdot x}{r|S^{n-1}|\cdot|x-y|^n}, \tag{74}$$

and

$$\partial_\nu E(x-y^*) = \frac{r^2 - y^*\cdot x}{r|S^{n-1}|\cdot|x-y^*|^n} = \frac{|y|^{n-2}(|y|^2 - y\cdot x)}{r^{n-1}|S^{n-1}|\cdot|x-y|^n}. \tag{75}$$

Then substituting those into (72), we get the *Poisson kernel*

$$\Pi(y,x) := \partial_\nu G_y(x) = \frac{r^2 - |y|^2}{r|S^{n-1}|\cdot|x-y|^n}. \tag{76}$$

and *Poisson's formula*

$$u(y) = \int_{\partial B_r} \Pi(y,x) g(x)\,\mathrm{d}^{n-1}x, \tag{77}$$

the latter being true if $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies (66). In particular, putting $u \equiv 1$ immediately gives

$$\int_{\partial B_r} \Pi(y,x)\,\mathrm{d}^{n-1}x = 1, \qquad y \in B_r. \tag{78}$$

One can check that the Poisson kernel for $n = 2$ is given by the same formula (76) with $n = 2$.

Remark 12. Let $u \geq 0$ be a harmonic function in some region that contains $\overline{B}_r$. Then from an application of the Poisson formula we infer the following *Harnack inequality*

$$u(y) \leq \frac{r^2 - |y|^2}{r|S^{n-1}|\cdot(r-|y|)^n}\int_{\partial B_r} u(x)\,\mathrm{d}^{n-1}x = \frac{r^{n-2}(r^2-|y|^2)}{(r-|y|)^n}u(0) = \frac{1-k^2}{(1-k)^n}u(0), \tag{79}$$

where $k = |y|/r$. A lower bound on $u(y)$ can also be obtained, leading to

$$\left(\frac{1}{1+k}\right)^{n-2}\frac{1-k}{1+k}\,u(0) = \frac{1-k^2}{(1+k)^n}u(0) \leq u(y) \leq \left(\frac{1}{1-k}\right)^{n-2}\frac{1+k}{1-k}\,u(0), \tag{80}$$

which are a slight quantitative improvement over (42).

To study differentiability properties of u given by the Poisson formula (77), we need to be able to differentiate under the integral sign. At the moment, the following simple rule will be sufficient for our needs.

Lemma 13 (Leibniz rule)**.** *Let X be a compact topological space, and let $f : X \times (a, b) \to \mathbb{R}$ be a (jointly) continuous function. We label the variables of f by $(x, t) \in X \times (a, b)$, and assume that $\frac{\partial f}{\partial t} : X \times (a, b) \to \mathbb{R}$ is also continuous. Let $T : \mathscr{C}(X) \to \mathbb{R}$ be a bounded linear map, i.e.,*

$$|Tu| \leq c\|u\|_{\mathscr{C}(X)}, \qquad u \in \mathscr{C}(X), \tag{81}$$

for some constant c. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Tf(\cdot, t) = T\frac{\partial f}{\partial t}(\cdot, t), \qquad t \in (a, b). \tag{82}$$

Proof. In this proof, we fix $t \in (a, b)$ once and for all. Then by the mean value theorem, for $x \in X$ and $h > 0$, there is some $0 \leq \theta(x, h) \leq h$ such that

$$\frac{f(x, t + h) - f(x, t)}{h} = f_t'(x, t + \theta(x, h)), \tag{83}$$

where we have abbreviated $f_t' = \frac{\partial f}{\partial t}$. For a fixed t, the left hand side is a continuous function of $x \in X$, hence we can apply T to both sides, and conclude that

$$\frac{Tf(\cdot, t + h) - Tf(\cdot, t)}{h} = T\big(\frac{f(\cdot, t + h) - f(\cdot, t)}{h}\big) = Tf_t'(\cdot, t + \theta(\cdot, h)). \tag{84}$$

The lemma would follow from linearity and boundedness of T, upon showing that

$$\sup_{x \in X} |f_t'(x, t + \theta(x, h)) - f_t'(x, t)| \to 0 \qquad \text{as} \quad h \to 0. \tag{85}$$

To see that this is true, let

$$\omega(x, s) = |f_t'(x, t + s) - f_t'(x, t)|, \tag{86}$$

which is a continuous function of $(x, s) \in X \times [0, \varepsilon)$ for some $\varepsilon > 0$, satisfying $\omega(x, 0) = 0$ for all $x \in X$. Since $0 \leq \theta(x, h) \leq h$, we have

$$\sup_{x \in X} |f_t'(x, t + \theta(x, h)) - f_t'(x, t)| \leq \sup_{(x,s) \in X \times [0,h]} \omega(x, s) =: w(h). \tag{87}$$

The function w is continuous because ω is continuous, and $w(0) = 0$ because $\omega(\cdot, 0) = 0$, thus (85) is established. □

We are now ready to tackle the Poisson formula.

Theorem 14 (Schwarz 1872)**.** *Let $g \in C(\partial B_r)$ and let u be given by (77). Then $u \in C^\infty(B_r)$, $\Delta u = 0$ in B_r, and $u(y) \to g(x)$ as $B_r \ni y \to x \in \partial B_r$.*

Proof. For any fixed $x \in \partial B_r$, the Poisson kernel $\Pi(y, x)$ is infinitely differentiable in y and $\Delta_y \Pi(y, x) = 0$ for $y \in B_r$, where Δ_y denotes the Laplace operator with respect to y. Indeed, an explicit calculation gives

$$\frac{\partial^2}{\partial y_i^2} \frac{1}{|x - y|^n} = \frac{n(n + 2)(x_i - y_i)^2}{|x - y|^{n+4}} - \frac{n}{|x - y|^{n+2}}, \tag{88}$$

and

$$\frac{\partial^2}{\partial y_i^2} \frac{|y|^2}{|x - y|^n} = \frac{n(n + 2)|y|^2(x_i - y_i)^2}{|x - y|^{n+4}} - \frac{n|y|^2 + 4ny_i(x_i - y_i)}{|x - y|^{n+2}} + \frac{2}{|x - y|^n}, \tag{89}$$

leading to

$$\Delta_y \frac{r^2 - |y|^2}{|x - y|^n} = \frac{2n(r^2 + |y|^2 - x \cdot y)}{|x - y|^{n+2}} - \frac{2n}{|x - y|^n} = 0, \qquad \text{for} \quad |x| = r. \tag{90}$$

Furthermore, observe that all partial derivatives $\frac{\partial^{k_1+\ldots+k_n}}{\partial^{k_1}y_1\cdots\partial^{k_n}y_n}\Pi(y,x)$ are continuous functions of $(y,x)\in B_r\times\partial B_r$, thus we can apply the Leibniz rule (Lemma 13) to the Poisson integral (77) repeatedly, and conclude that $u\in C^\infty(B_r)$, and that

$$\Delta u(y)=\Delta\int_{\partial B_r}\Pi(y,x)g(x)\,\mathrm{d}^{n-1}x=\int_{\partial B_r}g(x)\Delta_y\Pi(y,x)\,\mathrm{d}^{n-1}x=0, \tag{91}$$

for $y\in B_r$.

Now let $\hat{x}\in\partial B_r$. Then by (78) we have

$$g(\hat{x})=\int_{\partial B_r}\Pi(y,x)g(\hat{x})\,\mathrm{d}^{n-1}x, \tag{92}$$

and so

$$u(y)-g(\hat{x})=\int_{\partial B_r}\Pi(y,x)(g(x)-g(\hat{x}))\,\mathrm{d}^{n-1}x. \tag{93}$$

If $|x-\hat{x}|>\delta>0$ and $|y-\hat{x}|<\frac{\delta}{2}$, then $|x-y|>\frac{\delta}{2}$, so the function $x\mapsto\Pi(y,x)$ converges uniformly in $\partial B_r\setminus B_\delta(\hat{x})$ to 0 as $y\to\hat{x}$. For x close to $\hat{x}$, the continuity of g is enough to counteract the singularity of $x\mapsto\Pi(y,x)$ at $x=\hat{x}$, because this singularity is integrable uniformly in y as seen from (78). To formalize the argument, let $\delta>0$ be a constant to be adjusted later. Then by using the fact that $\Pi(y,x)$ is positive, we have

$$\begin{aligned}|u(y)-g(\hat{x})|&\leq\int_{\partial B_r}\Pi(y,x)|g(x)-g(\hat{x})|\,\mathrm{d}^{n-1}x\\&\leq\sup_{x\in\partial B_r\cap B_\delta(\hat{x})}|g(x)-g(\hat{x})|\int_{\partial B_r\cap B_\delta(\hat{x})}\Pi(y,x)\,\mathrm{d}^{n-1}x\\&\quad+\sup_{x\in\partial B_r\setminus B_\delta(\hat{x})}\Pi(y,x)\int_{\partial B_r\setminus B_\delta(\hat{x})}|g(x)-g(\hat{x})|\,\mathrm{d}^{n-1}x\\&\leq\sup_{x\in\partial B_r\cap B_\delta(\hat{x})}|g(x)-g(\hat{x})|+2\|g\|_{L^\infty}|\partial B_r|\sup_{x\in\partial B_r\setminus B_\delta(\hat{x})}\Pi(y,x).\end{aligned} \tag{94}$$

For any given $\varepsilon>0$, we can pick $\delta>0$ so small that the first term is smaller than ε. Then we choose y so close to $\hat{x}$ that the second term is smaller than ε. □

Remark 15. Poisson's formula generates a harmonic function u in B_r also when g is integrable with respect to the surface area, or even when $g\,\mathrm{d}^{n-1}x$ is merely a signed Borel measure of bounded variation. Then the *Herglotz theorem* states that by considering all signed Borel measures of bounded variation on the sphere ∂B_r, one recovers precisely the harmonic functions in B_r that are differences of two nonnegative harmonic functions in B_r. The limit of $u(y)$ as $B_r\ni y\to x\in\partial B_r$ can be studied as well. For instance, whenever g is continuous at x, we have $u(y)\to g(x)$. In general, depending on how y approaches x, the limit is related to various kinds of derivatives of the measure $g\,\mathrm{d}^{n-1}x$ with respect to the surface area measure.

One should not be deceived by the fact that Poisson's formula solves a seemingly simple problem. It is a very powerful tool in the study of harmonic functions.

Theorem 16 (Removable singularity)**.** *Let Ω be an open set, and let $z\in\Omega$. Assume that $u\in C^2(\Omega\setminus\{z\})$ is harmonic in $\Omega\setminus\{z\}$, and satisfies $u(x)=o(E(x-z))$ as $x\to z$. Then $u(z)$ can be defined so that $u\in C^2(\Omega)$ and $\Delta u=0$ in Ω.*

Proof. Without loss of generality, let us assume $z=0$ and $\overline{B}_r\subset\Omega$ with some $r>0$. Let $v\in C^2(B_r)$ satisfy $\Delta v=0$ in B_r and $v=u$ on ∂B_r. Of course, if u has a harmonic extension to B_r then it must be equal to v. For this to work, we need to show that $u=v$ in $B_r\setminus\{0\}$. By the maximum principle, we have $|v|\leq M_r$ in B_r, where $M_r=\sup_{x\in\partial B_r}|u(x)|$. Let $w=u-v$ and $\delta>0$. Then we have $\Delta w=0$ in $B_r\setminus B_\delta$ and $w=0$ on ∂B_r. We can say that

$|w| \leq |v| + |u| \leq M_r + M_\delta$ on ∂B_δ. At this point if we apply the maximum principle to w, we would only get $|w| \leq M_r + M_\delta$, which is not what we are after. Let us look at the case $n \geq 3$ first. We define the function $\phi(x) = (M_r + M_\delta)\delta^{n-2}/|x|^{n-2}$ for comparison purposes. We see that $\Delta\phi = 0$ in $B_r \setminus B_\delta$, $\phi \geq 0$ on ∂B_r and $\phi = M_r + M_\delta$ on ∂B_δ, i.e., $\pm w \leq \phi$ on the boundary of $B_r \setminus B_\delta$. Applying the maximum principle to $\pm w - \phi$ gives $|w(x)| \leq (M_r + M_\delta)\delta^{n-2}/|x|^{n-2}$ for $x \in B_r \setminus B_\delta$. Finally, for any fixed $x \in B_r$, sending $\delta \to 0$ and taking into account that $M_\delta = o(\delta^{2-n})$, we infer $|w(x)| = 0$. In the case $n = 2$, we can use the comparison function $\phi(x) = (M_r + M_\delta)\log(r/|x|)/\log(r/\delta)$. □

Exercise 12. Prove the *Hopf lemma*: Let $u \in C^2(B_r) \cap C(\overline{B}_r)$ be a function harmonic in B_r, which attains its maximum at $z \in \partial B_r$. Show that unless u is constant, there exists $c > 0$ such that $u(z) - u(zt) \geq (1-t)c$ for all $0 < t < 1$.

9. Converse to the mean value property

In this section, we prove that mean value property implies smoothness and harmonicity. After presenting a proof that is direct and elementary, we will hint at quicker proofs.

Lemma 17. *Let $u \in C(\Omega)$ be a function satisfying the mean value property for every ball whose closure is contained in Ω. Then $u \in C^1(\Omega)$, and for $\eta \in S^{n-1}$ and $\overline{B_r(y)} \subset \Omega$, we have*

$$\partial_\eta u(y) = \frac{1}{|B_r|}\int_{\partial B_r(y)} u\,\eta\cdot\nu = \frac{1}{|\partial B_r|}\int_{\partial B_r(y)} \partial_\eta u, \tag{95}$$

i.e., $\partial_\eta u$ satisfies the mean value property.

Proof. Let $y \in \Omega$, and let $\overline{B_r(y)} \subset \Omega$. Then for all small t, we have $\overline{B_r(y+\eta t)} \subset \Omega$. The mean value property gives

$$u(y+\eta t) - u(y) = \frac{1}{|B_r|}\int_{B_r(y)} (u(x+\eta t) - u(x))\,\mathrm{d}^n x. \tag{96}$$

Define $\partial^\pm B_r(y) = \{x \in \partial B_r(y) : (x-y)\cdot\eta \gtrless 0\}$, i.e., $\partial^+ B_r(y)$ is the positive half of $\partial B_r(y)$ with respect to the direction η, and $\partial^- B_r(y)$ is the negative half. Then the above integral can be decomposed as

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r(y)} (u(x+\eta t) - u(x))\,\mathrm{d}^n x &= \frac{1}{|B_r|}\int_{\partial^+ B_r(y)}\int_0^t u(x+\eta s)\,\eta\cdot\nu\,\mathrm{d}s\,\mathrm{d}^{n-1}x \\ &\quad - \frac{1}{|B_r|}\int_{\partial^- B_r(y)}\int_0^t u(x+\eta s)(-\eta\cdot\nu)\,\mathrm{d}s\,\mathrm{d}^{n-1}x,\end{aligned} \tag{97}$$

where ν is the outer unit normal to $\partial B_r(y)$, and the notation $\mathrm{d}^{n-1}x$ is meant to make it clear that the x-integration is over an $n-1$ dimensional surface. We can recombine the integrals and use uniform continuity to get

$$\begin{aligned}u(y+\eta t) - u(y) &= \frac{1}{|B_r|}\int_{\partial B_r(y)}\int_0^t u(x+\eta s)\,\eta\cdot\nu\,\mathrm{d}s\,\mathrm{d}^{n-1}x \\ &= \frac{t}{|B_r|}\int_{\partial B_r(y)} u(x)\,\eta\cdot\nu\,\mathrm{d}^{n-1}x + o(|t|),\end{aligned} \tag{98}$$

which proves the first equality in (95). As u is continuous, the integral over $B_r(y)$ depends on y continuously, hence $\partial_\eta u \in C(\Omega)$, implying that $u \in C^1(\Omega)$. The second equality in (95) follows from the divergence theorem. □

Exercise 13. Prove the preceding lemma by using Poisson's formula and maximum principles.

Theorem 18 (Koebe 1906)**.** *Let $u \in C(\Omega)$ be a function satisfying the mean value property for every ball whose closure is contained in Ω. Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$.*

Proof. The smoothness of u follows by induction from the preceding lemma. Also, Δu satisfies the mean value property, as it is a linear combination of derivatives of u. Applying the divergence theorem to this fact then reveals

$$\Delta u(y) = \frac{1}{|B_r|}\int_{B_r(y)} \nabla\cdot\nabla u = \frac{1}{|B_r|}\int_{\partial B_r(y)} \partial_\nu u. \tag{99}$$

The mean value property can be written as

$$u(y) = \frac{1}{|\partial B_r|}\int_{\partial B_r(y)} u = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} u(y+\xi r)\,\mathrm{d}^{n-1}\xi. \tag{100}$$

Then

$$0 = \frac{\mathrm{d}}{\mathrm{d}r}\int_{S^{n-1}} u(y+\xi r)\,\mathrm{d}^{n-1}\xi = \int_{S^{n-1}} \partial_r u(y+\xi r)\,\mathrm{d}^{n-1}\xi = \frac{1}{r^{n-1}}\int_{\partial B_r(y)} \partial_\nu u, \tag{101}$$

which completes the proof. □

Remark 19. An alternative argument proceeds by showing that for any $\varepsilon > 0$ there exists $\rho_\varepsilon \in \mathscr{D}(\mathbb{R}^n)$ such that $\rho_\varepsilon * u = u$ on a subset of Ω that covers Ω in the limit $\varepsilon \to 0$. Since the convolution of a test function with any locally integrable function is smooth, this implies that u is smooth in Ω. Let $\rho \in \mathscr{D}(\mathbb{R}^n)$ be a radially symmetric function, and let $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(\varepsilon^{-1}x)$ for $x \in \mathbb{R}^n$. Then for $y \in \Omega$ and $\varepsilon > 0$ sufficiently small, we have

$$(\rho_\varepsilon * u)(y) = \int \varepsilon^{-n}\rho(\varepsilon^{-1}x)u(y-x)\,\mathrm{d}x = \int \rho(\xi)u(y-\varepsilon\xi)\,\mathrm{d}\xi = u(y)\int\rho, \tag{102}$$

where in the last step we used the mean value property.

10. Derivative estimates and analyticity

Suppose that u is a harmonic function in Ω. Then from (95) we infer

$$|\partial_\eta u(y)| \leq \frac{1}{|B_r|}\int_{\partial B_r(y)} |u| \leq \frac{|\partial B_r|}{|B_r|}\sup_{\partial B_r(y)}|u| = \frac{n}{r}\sup_{\partial B_r(y)}|u|, \tag{103}$$

for $\eta \in S^{n-1}$ and $\overline{B_r(y)} \subset \Omega$. This means that a harmonic function tends to be flat towards the middle of the domain on which it is harmonic.

Exercise 14. Recall that a sequence $\{u_k\}$ converges to u *locally uniformly* (or *compactly*) in Ω if $u_k \to u$ uniformly in K for each compact $K \subset \Omega$. Prove that if $\{u_k\}$ is a sequence of functions harmonic in Ω, and if it converges to u locally uniformly in Ω, then u is harmonic in Ω, and $\partial_i u_k \to \partial_i u$ locally uniformly in Ω for each i.

If $u \geq 0$ in (103), we can use the mean value property to get

$$|\partial_\eta u(y)| \leq \frac{1}{|B_r|}\int_{\partial B_r(y)} u = \frac{|\partial B_r|}{|B_r|}u(y) = \frac{n}{r}u(y), \tag{104}$$

which is called a *differential Harnack inequality.* Liouville's theorem follows immediately: If u is nonnegative and entire harmonic, then at each point $y \in \mathbb{R}^n$, taking $r \to \infty$ in (104) implies that $\partial_\eta u(y) = 0$.

The differential Harnack inequality (104) can be integrated to get a Harnack inequality. Suppose that γ is a differentiable curve parameterized by arc length, with endpoints $x = \gamma(0)$

and $y = \gamma(\ell)$, such that each point on γ is at the distance greater than R from the boundary of Ω. Assume that $u > 0$ in Ω. Then we have

$$\frac{\mathrm{d}\log u(\gamma(t))}{\mathrm{d}t} = \frac{\gamma'(t)\cdot\nabla u(\gamma(t))}{u(\gamma(t))}, \qquad 0 \leq t \leq \ell. \tag{105}$$

Integrating this, we get

$$|\log u(x) - \log u(y)| \leq \int_0^\ell \big|\frac{\gamma'(t)\cdot\nabla u(\gamma(t))}{u(\gamma(t))}\big|\,\mathrm{d}t \leq \frac{n\ell}{R}, \tag{106}$$

which implies

$$e^{-n\ell/R} \leq \frac{u(x)}{u(y)} \leq e^{n\ell/R}. \tag{107}$$

The essence of Harnack inequalities is the fact that the ratio $\frac{u(x)}{u(y)}$ cannot be too large if the influence from the boundary is weak relative to the interaction between x and y.

We can repeatedly apply (103) to derive estimates on higher derivatives.

Lemma 20. *Let u be harmonic in Ω, and let $\overline{B_r(y)} \subset \Omega$. Then*

$$|\partial^\alpha u(y)| \leq |\alpha|!\big(\frac{ne}{r}\big)^{|\alpha|} \sup_{B_r(y)} |u|. \tag{108}$$

Proof. Let $\rho = \frac{r}{|\alpha|}$ and let β be a multi-index with $|\beta| = |\alpha| - 1$. Then since all derivatives of a harmonic function are also harmonic, from (103) we have

$$|\partial^\alpha u(y)| \leq \frac{n}{\rho} \sup_{\partial B_\rho(y)} |\partial^\beta u|. \tag{109}$$

We can estimate the derivative $\partial^\beta u$ appearing in the right hand side by the same procedure, decreasing the order of derivatives again by one. We continue this process until we get no derivatives in the right hand side, and get

$$|\partial^\alpha u(y)| \leq \big(\frac{n}{\rho}\big)^{|\alpha|} \sup_{B_r(y)} |u| = \big(\frac{n|\alpha|}{r}\big)^{|\alpha|} \sup_{B_r(y)} |u|. \tag{110}$$

The estimate (108) follows from here upon using the elementary inequality $k^k \leq k!e^k$, which can be seen for instance from the convergent series $e^k = 1 + k + \ldots + \frac{k^k}{k!} + \ldots$. □

In particular, harmonic functions are analytic, because of the following. Recall that $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ is simply the ball centred at 0 of radius r.

Lemma 21. *Let $u \in C^\infty(B_r)$ be such that*

$$|\partial^\alpha u(x)| \leq M|\alpha|!\big(\frac{c}{r-|x|}\big)^{|\alpha|}, \qquad x \in B_1, \tag{111}$$

for some constants $c > 0$ and $M > 0$. Then the Maclaurin series

$$u(x) = \sum_\alpha \frac{\partial^\alpha u(0)}{\alpha!} x^\alpha \equiv \sum_{\alpha_1,\ldots,\alpha_n \geq 0} \frac{\partial^{\alpha_1}\ldots\partial^{\alpha_n} u(0)}{\alpha_1!\ldots\alpha_n!} x_1^{\alpha_1}\ldots x_n^{\alpha_n}, \tag{112}$$

absolutely converges if $|x| < \frac{r}{1+cn}$. Here $\alpha! = \alpha_1!\ldots\alpha_n!$.

Proof. Given $z \in B_r$, consider the function $f(t) = u(zt)$. Taylor's theorem tells us

$$u(z) = f(1) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} + \frac{f^{(m)}(s)}{m!}, \tag{113}$$

where $0 \leq s \leq 1$. Let us compute the derivatives of f. We have

$$\begin{aligned} f'(t) &= (z_1\partial_1 + \ldots + z_n\partial_n)u(zt), \\ f''(t) &= (z_1\partial_1 + \ldots + z_n\partial_n u)^2 u(zt), \ldots \\ f^{(k)}(t) &= (z_1\partial_1 + \ldots + z_n\partial_n)^k u(zt) \\ &= \sum_{\alpha_1 \ldots + \alpha_n = k} \frac{k!}{\alpha_1! \ldots \alpha_n!} z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} u(zt) \\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} z^\alpha \partial^\alpha u(zt), \end{aligned} \tag{114}$$

by the multinomial theorem, so

$$u(z) = \sum_{|\alpha|<m} \frac{\partial^\alpha u(0)}{\alpha!} z^\alpha + \underbrace{\frac{(z_1\partial_1 + \ldots + z_n\partial_n)^m u(sz)}{m!}}_{R_m}, \tag{115}$$

with $0 \leq s \leq 1$. We can estimate the remainder term by

$$|R_m| \leq M\big(\frac{c}{r-|z|}\big)^m n^m |z|^m = M\big(\frac{cn|z|}{r-|z|}\big)^m, \tag{116}$$

which tends to 0 if $\frac{cn|z|}{r-|z|} < 1$, i.e., $|z| < \frac{r}{1+cn}$. □

11. Sequences of harmonic functions

In 1887, Axel Harnack proved two fundamental convergence theorems for sequences of harmonic functions. The first of them concerns uniform convergence and can be thought of as an analogue of the Weierstrass convergence theorem from complex analysis.

Theorem 22 (Harnack's first theorem). *Let Ω be a bounded domain, and let $\{u_j\}$ be a sequence of harmonic functions in Ω.*

a) If $\{u_j\}$ converges uniformly on $\partial\Omega$, then it converges uniformly in $\overline{\Omega}$.

b) If $u_j \to u$ locally uniformly in Ω, then u is harmonic in Ω. Furthermore, $\partial^\alpha u_j \to \partial^\alpha u$ locally uniformly in Ω for each multi-index α, i.e., $u_j \to u$ in $C^\infty(\Omega)$.

Proof. a) It follows from the maximum principle that

$$\sup_\Omega |u_j - u_k| \leq \sup_{\partial\Omega} |u_j - u_k|, \tag{117}$$

which means that $\{u_j\}$ is Cauchy in the topology of uniform convergence in $\overline{\Omega}$, hence it converges uniformly in $\overline{\Omega}$.

b) For any $\overline{B_r(y)} \subset \Omega$ and any j, we have

$$u_j(y) = \frac{1}{|B_r|} \int_{B_r(y)} u_j. \tag{118}$$

By the locally uniform convergence, $u_j(y) \to u(y)$ and $\int_{B_r(y)} u_j \to \int_{B_r(y)} u$, which implies that u satisfies the mean value property for every ball whose closure is in Ω. Hence u is harmonic. Let $K \Subset \Omega$ be a compact set. Then there exists a compact set $K' \Subset \Omega$ such that $K \subset K'$ and $r = \mathrm{dist}(K, \partial K') > 0$. Now the derivative estimates for harmonic functions (Lemma 20) gives

$$\sup_K |\partial^\alpha(u_j - u)| \leq |\alpha|!\big(\frac{ne}{r}\big)^{|\alpha|} \sup_{K'} |u_j - u|, \tag{119}$$

which completes the proof. □

Before stating the second theorem of Harnack which deals with nondecreasing sequences of harmonic functions, we prove a generalized version of the Harnack inequality.

Lemma 23 (Harnack inequality)**.** *Let Ω be a domain (i.e., connected open set), and let $K \Subset \Omega$ be its compact subset. Then there exists a constant $C > 0$, possibly depending on K, such that for any harmonic and nonnegative function u in Ω, we have*

$$u(x) \leq Cu(y) \qquad x, y \in K. \tag{120}$$

Proof. It suffices to prove the inequality for strictly positive harmonic functions, since if $u \geq 0$ then for any $\varepsilon > 0$ we would have

$$u(x) + \varepsilon \leq Cu(y) + C\varepsilon \qquad x, y \in K, \tag{121}$$

and sending $\varepsilon \to 0$ would establish the claim.

Recall the primitive Harnack inequality (Lemma 3 and Remark 12): If $\overline{B_{2r}(y)} \subset \Omega$ and $x \in B_r(y)$ then $u(x) \leq 2^n u(y)$. The idea is to piece together primitive Harnack inequalities to connect any pair of points in Ω. One way of doing this was discussed in the derivation of (107). Here we will use a slightly different approach. For $x, y \in \Omega$, define

$$s(x, y) = \sup \big\{ \frac{u(x)}{u(y)} : u > 0,\, \Delta u = 0 \text{ in } \Omega \big\}. \tag{122}$$

First, let us prove that $s(x, y)$ is finite for any $x, y \in \Omega$. Fix $y \in \Omega$, and let $\Sigma = \{x \in \Omega : s(x, y) < \infty\}$. Obviously $y \in \Sigma$, so Σ is nonempty. If $x \in \Sigma$, then $u \leq 2^n u(x)$ in a small ball centred at x, so Σ is open. Moreover, Σ is relatively closed in Ω, because if $\Sigma \ni x_j \to x \in \Omega$ then for sufficiently large j we would have $u(x) \leq 2^n u(x_j)$. We conclude that $\Sigma = \Omega$.

Let K be compact subset of Ω, and let $r = \frac{1}{3}\text{dist}(K, \partial\Omega)$. Then we can cover $K \times K$ by finitely many sets of the form $B_r(a) \times B_r(b)$, with $(a, b) \in K \times K$. This means that for any pair $(x, y) \in K \times K$, there is a pair (a, b) taken from a finite collection, say P, such that $x \in B_r(a)$ and $y \in B_r(b)$. We immediately have $u(x) \leq 2^n u(a)$ and $u(b) \leq 2^n u(y)$, which implies that $u(x) \leq 2^{2n} \big(\max_{(a,b) \in P} s(a, b) \big) u(y)$. □

The second theorem does not have a good counterpart for holomorphic functions, as it relies on the order structure of $\mathbb{R}$.

Theorem 24 (Harnack's second theorem)**.** *Let Ω be a domain, and let $u_1 \leq u_2 \leq \ldots$ be a nondecreasing sequence of harmonic functions in Ω. Then either*

- *$u_j(x) \to \infty$ for each $x \in \Omega$, or*
- *$\{u_j\}$ converges locally uniformly in Ω.*

Proof. Suppose that $u_j(y) \leq M$ for some $y \in \Omega$ and $M < \infty$. Obviously, $u_j(y)$ is convergent. Let K be a compact subset of Ω, and without loss of generality, assume that $y \in K$. Then since $u_{j+k} - u_j \geq 0$ for $k > 0$, by Harnack inequality there exists $C > 0$ such that for any $x \in K$ we have

$$u_{j+k}(x) - u_j(x) \leq C(u_{j+k}(y) - u_j(y)), \tag{123}$$

which implies that $\{u_j\}$ converges uniformly in K. Then by Harnack's first theorem, the limit is harmonic in the interior of K. □

Now we study sequential compactness of bounded families of harmonic functions, in the topology of locally uniform convergence. Such a compactness is customarily called *normaility.* First we recall the all important Arzelà-Ascoli theorem, in a form convenient for our purposes.

Theorem 25 (Arzelà-Ascoli)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $f_j : \Omega \to \mathbb{R}$ be a sequence that is locally equicontinuous and locally equibounded. Then there is a subsequence of $\{f_j\}$ that converges locally uniformly.*

That the sequence $\{f_j\}$ is *locally equibounded* means that for any compact set $K \subset \Omega$ one has $\sup_j \sup_K |f_j| < \infty$. Similarly, that the sequence $\{f_j\}$ is *locally equicontinuous* means that for any compact set $K \subset \Omega$ the sequence $\{f_j\}$ is (uniformly) equicontinuous on K. If $\{f_j\}$ is a sequence of harmonic functions, then the equicontinuity condition can be dropped from the Arzelà-Ascoli theorem, because we can bound derivatives of a harmonic function by how large the function itself is. This is an analogue of Montel's theorem in complex analysis.

Theorem 26. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\{f_j\}$ be a locally equibounded sequence of harmonic functions in Ω. Then there is a subsequence of $\{f_j\}$ that converges locally uniformly.*

Proof. In view of the Arzelà-Ascoli theorem, it suffices to show local equicontinuity of $\{f_j\}$. We will prove here that $\{f_j\}$ is equicontinuous on any closed ball $\overline{B} \subset \Omega$, and the general case follows by a covering argument. Let $B = B_\rho(y)$ and $B' = B_{\rho+r}(y)$ be two concentric balls such that $\overline{B'} \subset \Omega$ and $r > 0$. Then the gradient estimate (103) gives

$$|\nabla f_j(x)| \leq \frac{n}{r} \max_{\partial B_r(x)} |f_j|, \quad \text{for} \quad x \in B, \qquad \text{hence} \qquad \sup_B |\nabla f_j| \leq \frac{n}{r} \sup_{B'} |f_j|.$$

For $x, z \in B$, we have

$$|f_j(z) - f_j(x)| \leq |z - x| \cdot \frac{n}{r} \sup_{B'} |f_j|.$$

Since $\sup_{B'} |f_j|$ is bounded uniformly in j, the sequence $\{f_j\}$ is equicontinuous on B. □