THE DIRICHLET PROBLEM

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Abstract. We present here two approaches to the Dirichlet problem: The classical method of subharmonic functions that culminated in the works of Perron and Wiener, and the more modern Sobolev space approach tied to calculus of variations.

Contents

1. Brief history

Given a domain $\Omega \subset \mathbb{R}^n$ and a function $g : \partial\Omega \to \mathbb{R}$, the *Dirichlet problem* is to find a function u satisfying

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In the previous set of notes, we established that uniqueness holds if Ω is bounded and g is continuous. We have also seen that the Dirichlet problem has a solution if Ω is a ball.

The Dirichlet problem turned out to be fundamental in many areas of mathematics and physics, and the efforts to solve this problem led directly to many revolutionary ideas in mathematics. The importance of this problem cannot be overstated.

The first serious study of the Dirichlet problem on general domains with general boundary conditions was done by George Green in his *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, published in 1828. He reduced the problem into a problem of constructing what we now call Green's functions, and argued that Green's function exists for any domain. His methods were not rigorous by today's standards, but the ideas were highly influential in the subsequent developments. It should be noted that George Green had basically no formal schooling when he wrote the Essay, and most probably he never knew the real importance of his discovery, as the Essay went unnoticed by the community until 1845, four years after Green's death, when William Thomson rediscovered it.

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The next idea came from Gauss in 1840. He noticed that given a function ρ on $\partial\Omega$, the so-called single layer potential

$$u(y) = (V\rho)(y) \equiv \int_{\partial\Omega} E_y \rho, \tag{2}$$

is harmonic in Ω, and hence that if we find ρ satisfying $V\rho = g$ on $\partial\Omega$, the Dirichlet problem would be solved. Informally, we want to arrange electric charges on the surface $\partial\Omega$ so that the resulting electric potential is equal to g on $\partial\Omega$. If we imagine that $\partial\Omega$ is made of a good conductor, then in the absence of an external field, the equilibrium configuration of charges on the surface will be the one that produces constant potential throughout $\partial\Omega$. The same configuration also minimizes the electrostatic energy

$$E(\rho) = \frac{1}{2}\int_{\partial\Omega} \rho V\rho, \tag{3}$$

among all ρ such that the net charge $\int_{\partial\Omega}\rho$ is fixed. In order to solve $V\rho = g$, we imagine that there is some external electric field whose potential at the surface coincides with $-g$. The equilibrium configuration in this case would satisfy $V\rho - g = \text{const}$, and minimize the energy

$$E(\rho) = \frac{1}{2}\int_{\partial\Omega} \rho V\rho - \int_{\partial\Omega} g\rho, \tag{4}$$

among all ρ such that the net charge is fixed. Then we would have $V(\rho - \rho') = g$ for some ρ' satisfying $V\rho' = \text{const}$, or more directly, we can simply add a suitable constant to $u = V\rho$ to solve the Dirichlet problem. Gauss did not prove the existence of a minimizer to (4), but he remarked that it was obvious.

Around 1847, that is just after Green's work became widely known, William Thomson (Lord Kelvin) and Gustav Lejeune-Dirichlet suggested to minimize the energy

$$E(u) = \int_{\Omega} |\nabla u|^2, \tag{5}$$

subject to $u|_{\partial\Omega} = g$. Note that by Green's first identity we have

$$E(u) = \int_{\partial\Omega} u\partial_\nu u - \int_{\Omega} u\Delta u, \tag{6}$$

which explains why $E(u)$ can be considered as the energy of the configuration, since in view of Green's formula

$$u(y) = \int_{\Omega} E_y \Delta u + \int_{\partial\Omega} u\partial_\nu E_y - \int_{\partial\Omega} E_y \partial_\nu u, \tag{7}$$

$\partial_\nu u$ is the surface charge density, and $-\Delta u$ is the volume charge density, that produce the field u. This and other considerations seemed to show that the Dirichlet problem is equivalent to minimizing the energy $E(u)$ subject to $u|_{\partial\Omega} = g$. Moreover, since $E(u) \geq 0$ for any u, the existence of u minimizing $E(u)$ appeared to be obvious. Riemann called these two statements the *Dirichlet principle*, and used it to prove his fundamental mapping theorem, in 1851. However, starting around 1860, the Dirichlet principle in particular and calculus of variations at the time in general went under serious scrutiny, most notably by Karl Weierstrass and Riemann's former student Friedrich Prym. Weierstrass argued that even if E is bounded from below, it is possible that the infimum is never attained by an admissible function, in which case there would be no admissible function that minimizes the energy. He backed his reasoning by an explicit example of an energy that has no minimizer. In 1871, Prym constructed a striking example of a continuous function g on the boundary of a disk, such that there is not a single function u with finite energy that equals g on the boundary. This makes it impossible even to talk about a minimizer since all functions with the correct boundary condition would have infinite energy. We will see a similar example constructed by Hadamard in §4, Example 13. Here we look at Weierstrass' example.

Example 1 (Weierstrass 1870)**.** Consider the problem of minimizing the energy

$$Q(u) = \int_I x^2 |u'(x)|^2 \,\mathrm{d}x, \tag{8}$$

for all $u \in C(\overline{I})$ with piecewise continuous derivatives in I, satisfying the boundary conditions $u(-1) = 0$ and $u(1) = 1$, where $I = (-1, 1)$. The infimum of E over the admissible functions is 0, because obviously $E \geq 0$ and for the function

$$v(x) = \begin{cases} 0 & \text{for } x < 0, \\ x/\delta & \text{for } 0 < x < \delta, \\ 1 & \text{for } x > \delta, \end{cases} \tag{9}$$

we have $E(v) = \frac{\delta}{3}$, which can be made arbitrarily small by choosing $\delta > 0$ small. However, there is no admissible function u for which $E(u) = 0$, since this would mean that $u(x) = 0$ for $x < 0$ and $u(x) = 1$ for $x > 0$.

Exercise 1 (Courant)*.* Consider the problem of minimizing the energy

$$Q(u) = \int_I \left(1 + |u'(x)|^2\right)^{\frac{1}{4}} \,\mathrm{d}x, \tag{10}$$

for all $u \in C^1(I) \cap C(\overline{I})$ satisfying $u(0) = 0$ and $u(1) = 1$, where $I = (0, 1)$. Show that the infimum of Q over the admissible functions is 1, but this value is not assumed by any admissible function.

Now that the Dirichlet principle is not reliable anymore, it became an urgent matter to solve the Dirichlet problem to "rescue" the Riemann mapping theorem. By 1870, Weierstrass' former student Hermann Schwarz had largely succeeded in achieving this goal. He solved the Dirichlet problem on polygonal domains by an explicit formula, and used an iterative approximation process to extend his results to an arbitrary planar region with piecewise analytic boundary. His approximation method is now known as the *Schwarz alternating method*, and is one of the popular methods to solve boundary value problems on a computer.

The next advance was Carl Neumann's work of 1877, that was based on the earlier work of August Beer from 1860. The idea was similar to Gauss', but instead of the single layer potential, Beer suggested the use of the double layer potential

$$u(y) = (K\mu)(y) \equiv \int_{\partial\Omega} \mu \partial_\nu E_y. \tag{11}$$

The function u is automatically harmonic in Ω, and the requirement $u|_{\partial\Omega} = g$ is equivalent to the integral equation $(1-2K)\mu = 2g$ on the boundary. This equation was solved by Neumann in terms of the series

$$(1 - 2K)^{-1} = 1 + 2K + (2K)^2 + \ldots, \tag{12}$$

which bears his name now. Neumann showed that the series converges if Ω is a 3 dimensional convex domain whose boundary does not consist of two conical surfaces. The efforts to solve the equation $(1 - 2K)\mu = 2g$ in cases the above series does not converge, led Ivar Fredholm to his discovery of Fredholm theory in 1900.

Since the analyticity or convexity conditions on the boundary seemed to be rather artificial, the search was still on to find a good method to solve the general Dirichlet problem. Then in 1887, Henri Poincaré published a paper introducing a very flexible method with far reaching consequences. Poincaré started with a subharmonic function that has the correct boundary values, and repeatedly solved the Dirichlet problem on small balls to make the function more and more nearly harmonic. He showed that the process converges if the succession of balls is chosen carefully, and produces a harmonic function in the interior. Moreover, this harmonic function assumes correct boundary values, if each point on the boundary of the domain can be touched from outside by a nontrivial sphere. The process is now called Poincaré's *sweeping out process* or the *balayage method*.

Poincaré's work made the Dirichlet problem very approachable, and invited further work on weakening the conditions on the boundary. For instance, it led to the work of William Fogg Osgood, published in 1900, in which the author establishes solvability of the Dirichlet problem in very general planar domains. While the situation was quite satisfactory, there had essentially been no development as to the validity of the original Dirichlet principle, until 1899, when David Hilbert gave a rigorous justification of the Dirichlet principle under some assumptions on the boundary of the domain. This marked the beginning of a major program to put calculus of variations on a rigorous foundation.

During that period it was generally believed that the assumptions on the boundary of the domain that seemed to be present in all available results were due to limitations of the methods employed, rather than being inherent in the problem. It was Stanisław Zaremba who first pointed out in 1911 that there exist regions in which the Dirichlet problem is not solvable, even when the boundary condition is completely reasonable.

Example 2 (Zaremba 1911)**.** Let $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disk, and consider the domain $\Omega = \mathbb{D} \setminus \{0\}$. The boundary of Ω consists of the circle $\partial\mathbb{D}$ and the point $\{0\}$. Consider the Dirichlet problem $\Delta u = 0$ in Ω, with the boundary conditions $u \equiv 0$ on $\partial\mathbb{D}$ and $u(0) = 1$. Suppose that there exists a solution. Then u is harmonic in Ω, and continuous in $\mathbb{D}$ with $u(0) = 1$. Since u is bounded in Ω, one can extend u continuously to $\mathbb{D}$ so that the resulting function is harmonic in $\mathbb{D}$. By uniqueness for the Dirichlet problem in $\mathbb{D}$, the extension must identically be equal to 0, because $u \equiv 0$ on $\partial\mathbb{D}$. However, this contradicts the fact that u is continuous in $\mathbb{D}$ with $u(0) = 1$. Hence there is no solution to the original problem. In other words, the boundary condition at $x = 0$ is simply "ignored".

One could argue that Zaremba's example is not terribly surprising because the boundary point 0 is an isolated point. However, in 1913, Henri Lebesgue produced an example of a 3 dimensional domain whose boundary consists of a single connected piece. This example will be studied in §3, Example 10. The time period under discussion is now 1920's, which saw intense developments in the study of the Dirichlet problem, then known as potential theory, powered by the newly founded Lebesgue integration theory and functional analytic point of view. Three basic approaches were most popular: Poincaré-type methods which use subharmonic functions, integral equation methods based on potential representations of harmonic functions, and finally, variational methods related to minimizing the Dirichlet energy. While the former two would still be considered as part of potential theory, the third approach has since separated because of its distinct Hilbert space/variational flavour. In what follows, we will study two specific methods in detail.

2. Perron's method

In this section, we will discuss the method discovered by Oskar Perron in 1923, as a simpler replacement of the Poincaré process. Recall that we want to solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{13}$$

In what follows, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, and that $g : \partial\Omega \to \mathbb{R}$ is a bounded function. Recall that a continuous function $u \in C(\Omega)$ is called *subharmonic* in Ω, if for any $y \in \Omega$, there exists $r^* > 0$ such that

$$u(y) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u, \qquad 0 < r < r^*. \tag{14}$$

Let us denote by $\mathfrak{Sub}(\Omega)$ the set of subharmonic functions on Ω. The following properties will be useful.

- If $u \in \mathfrak{Sub}(\Omega)$ and if $u(z) = \sup_{\Omega} u$ for some $z \in \Omega$, then u is constant.
- If $u \in \mathfrak{Sub}(\Omega) \cap C(\overline{\Omega})$, $v \in C(\overline{\Omega})$ is harmonic in Ω, and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.
- If $u_1, u_2 \in \mathfrak{Sub}(\Omega)$ then $\max\{u_1, u_2\} \in \mathfrak{Sub}(\Omega)$.
- If $u \in \mathfrak{Sub}(\Omega)$ and if $\bar{u} \in C(\Omega)$ satisfies $\Delta \bar{u} = 0$ in B and $\bar{u} = u$ in $\Omega \setminus B$ for some $B \subset \Omega$, then $\bar{u} \in \mathfrak{Sub}(\Omega)$.

The first two properties are simply the strong and weak maximum principles. The third property is clear from

$$u_i(y) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u_i \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} \max\{u_1, u_2\}, \qquad i = 1, 2. \tag{15}$$

For the last property, we only need to check (14) for $y \in \partial B$, as

$$\bar{u}(y) = u(y) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} \bar{u}. \tag{16}$$

To proceed further, we define the *Perron (lower) family*

$$S_g = \{v \in \mathfrak{Sub}(\Omega) \cap C(\overline{\Omega}) : v|_{\partial\Omega} \leq g\}, \tag{17}$$

and the *Perron (lower) solution*

$$u(x) = (P_\Omega g)(x) = \sup_{v \in S_g} v(x), \qquad x \in \Omega. \tag{18}$$

Any constant function c satisfying $c \leq g$ is in S_g, so $S_g \neq \varnothing$. Moreover, any $v \in S_g$ satisfies $v \leq \sup_{\partial\Omega} g$, hence the Perron solution u is well-defined. We will show that the Perron solution is a solution of the Dirichlet problem, under some mild regularity assumptions on the boundary of Ω. Before doing so, let us perform a consistency check. Suppose that $\Delta w = 0$ in Ω and $w|_{\partial\Omega} = g$. Then obviously $w \in S_g$. Also, the weak maximum principle shows that any $v \in S_g$ satisfies $v \leq w$ pointwise. Therefore we must have $u = w$.

Theorem 3 (Perron 1923)**.** *For the Perron solution $u = P_\Omega g$, we have $\Delta u = 0$ in Ω.*

Proof. Let $B_r(x)$ be a nonempty open ball whose closure is in Ω, and let $\{u_k\} \subset S_g$ be a sequence satisfying $u_k(x) \to u(x)$ as $k \to \infty$. Without loss of generality, we can assume that the sequence is nondecreasing, e.g., by replacing u_k by $\max\{u_1, \ldots, u_k\}$. For each k, let $\bar{u}_k \in C(\Omega)$ be the function harmonic in $B_r(x)$ which agrees with u_k in $\Omega \setminus B_r(x)$. We have $u_k \leq \bar{u}_k$, and $\bar{u}_k \in S_g$ hence $\bar{u}(x) \leq u(x)$, so $\bar{u}_k(x) \to u(x)$ as well. The sequence $\{\bar{u}_k\}$ is also nondecreasing, so by Harnack's second convergence theorem, there exists a harmonic function $\bar{u}$ in $B_r(x)$ such that $\bar{u}_k \to \bar{u}$ locally uniformly in $B_r(x)$. In particular, we have $\bar{u}(x) = u(x)$.

We want to show that $u = \bar{u}$ in $B_r(x)$, which would then imply that u is harmonic in Ω. Pick $y \in B_r(x)$, and let $\{\tilde{u}_k\} \subset S_g$ be a sequence satisfying $\tilde{u}_k(y) \to u(y)$. Without loss of generality, we can assume that the sequence is nondecreasing, that $\bar{u}_k \leq \tilde{u}_k$, and that $\tilde{u}_k$ is harmonic in $B_r(x)$. Again by Harnack's theorem, there exists a harmonic function $\tilde{u}$ in $B_r(x)$ such that $\tilde{u}_k \to \tilde{u}$ locally uniformly in $B_r(x)$, and we have $\tilde{u}(y) = u(y)$. Because of the arrangement $\bar{u}_k \leq \tilde{u}_k$, we get $\bar{u} \leq \tilde{u}$ in $B_r(x)$, and in addition taking into account that $\tilde{u}_k \leq u$ and that $\bar{u}_k(x) \to u(x)$, we infer $\tilde{u}(x) = u(x)$. So $\bar{u} - \tilde{u}$ is harmonic and nonpositive in $B_r(x)$, while $\bar{u}(x) - \tilde{u}(x) = 0$. Then the strong maximum principle gives $\bar{u} = \tilde{u}$ in $B_r(x)$, which implies that $\bar{u}(y) = u(y)$. As $y \in B_r(x)$ was arbitrary, $u = \bar{u}$ in $B_r(x)$. $\square$

Now we need to check if u satisfies the required boundary condition $u|_{\partial\Omega} = g$. Let $z \in \partial\Omega$, and let us try to imagine what can go wrong so that $u(x) \not\to g(z)$ as $x \to z$. It is possible that $\liminf_{x\to z} u(x) < g(z)$, or $\limsup_{x\to z} u(x) > g(z)$, or both. To rule out the first scenario, it suffices to show that there is a sequence $\{w_k\} \in S_g$ such that $w_k(z) \to g(z)$. Indeed, since $u \geq w_k$ pointwise, we would have $\liminf_{x\to z} u(x) \geq w_k(z)$ for each k. The existence of such a sequence $\{w_k\}$ means, in a certain sense, that the domain Ω is able to support a sufficiently rich family of subharmonic functions. In a similar fashion, to rule out the second scenario, we need to have a sufficiently rich family of superharmonic functions, and as superharmonic functions are simply the negatives of subharmonic functions, it turns out that both scenarios can be handled by the same method. We start by introducing the concept of a barrier.

Definition 4. A function $\varphi \in C(\overline{\Omega})$ is called a *barrier for* Ω *at* $z \in \partial\Omega$ if

- $\varphi \in \mathfrak{Sub}(\Omega)$,
- $\varphi(z) = 0$,
- $\varphi < 0$ on $\partial\Omega \setminus \{z\}$.

We call the boundary point $z \in \partial\Omega$ *regular* if there is a barrier for Ω at $z \in \partial\Omega$.

Lemma 5. *Let $z \in \partial\Omega$ be a regular point, and let g be continuous at z. Then for any given $\varepsilon > 0$, there exists $w \in S_g$ such that $w(z) \geq g(z) - \varepsilon$.*

Proof. Let $\varepsilon > 0$, and let φ be a barrier at z. Then there exists $\delta > 0$ such that $|g(x)-g(z)| < \varepsilon$ for $x \in \partial\Omega \cap B_\delta(z)$. Choose $M > 0$ so large that $M\varphi(x) + 2\|g\|_\infty < 0$ for $x \in \partial\Omega \setminus B_\delta(z)$, and consider the function $w = M\varphi + g(z) - \varepsilon$. Obviously, $w \in \mathfrak{Sub}(\Omega) \cap C(\overline{\Omega})$ and $w(z) = g(z) - \varepsilon$. Moreover, we have

$$M\varphi(x) + g(z) - \varepsilon < M\varphi(x) + g(x) \leq g(x), \qquad x \in \partial\Omega \cap B_\delta(z), \tag{19}$$

and

$$M\varphi(x) + g(z) - \varepsilon < -2\|g\|_\infty + g(z) \leq g(x), \qquad x \in \partial\Omega \setminus B_\delta(z), \tag{20}$$

which imply that $w \in S_g$. □

Exercise 2. Why is regularity of a boundary point a local property? In other words, if $z \in \partial\Omega$ is regular, and if Ω' is a domain that coincides with Ω in a neighbourhood of z (hence in particular $z \in \partial\Omega'$), then can you conclude that z is also regular as a point of $\partial\Omega'$?

Exercise 3. Show that if the Dirichlet problem in Ω is solvable for any boundary condition $g \in C(\partial\Omega)$, then each $z \in \partial\Omega$ is a regular point.

The following theorem implies the converse to the preceding exercise: If all boundary points are regular, then the Dirichlet problem is solvable for any $g \in C(\partial\Omega)$.

Theorem 6 (Perron 1923)**.** *Assume that $z \in \partial\Omega$ is a regular point, and that g is continuous at z. Then we have $u(x) \to g(z)$ as $\Omega \ni x \to z$.*

Proof. By Lemma 5, for any $\varepsilon > 0$ there exists $w \in S_g$ such that $w(z) \geq g(z) - \varepsilon$. By definition, we have $u \geq w$ in Ω. This shows that

$$\liminf_{\Omega\ni x\to z} u(x) \geq g(z) - \varepsilon, \tag{21}$$

and as $\varepsilon > 0$ was arbitrary, the same relation is true with $\varepsilon = 0$. On the other hand, again by Lemma 5, for any $\varepsilon > 0$ there exists $w \in S_{-g}$ such that $w(z) \geq -g(z) - \varepsilon$. Let $v \in S_g$. Then $v + w \in \mathfrak{Sub}(\Omega) \cap C(\overline{\Omega})$ and $v + w \leq 0$ on $\partial\Omega$. This means that $v \leq -w$ in Ω. Since v is an arbitrary element of S_g, the same inequality is true for u, hence

$$\limsup_{\Omega\ni x\to z} u(x) \leq g(z) + \varepsilon, \tag{22}$$

and as $\varepsilon > 0$ was arbitrary, the same relation is true with $\varepsilon = 0$. □

Exercise 4. Let us modify the definition of a barrier (Definition 4) by allowing $\varphi \in C(\Omega)$ and replacing the condition after the third bullet point therein by

$$\limsup_{\Omega \ni x \to y} \varphi(x) < 0 \qquad \text{for each} \quad y \in \partial\Omega \setminus \{z\}. \tag{23}$$

Show that Theorem 6 is still valid when the regularity concept is accordingly modified. ⊘

Corollary 7. *Green's function exists for the domain Ω if each point of $\partial\Omega$ is regular.*

3. Boundary regularity

It is of importance to derive simple criteria for a boundary point to admit a barrier. The following is referred to as *Poincaré's criterion* or the *exterior sphere condition.*

Theorem 8 (Poincaré 1887)**.** *Suppose that $B_r(y) \cap \Omega = \varnothing$ and $\overline{B_r(y)} \cap \partial\Omega = \{z\}$, with $r > 0$. Then z is a regular point.*

Proof. For $n \geq 3$, we claim that

$$\varphi(x) = \frac{1}{|x-y|^{n-2}} - \frac{1}{r^{n-2}}, \qquad x \in \overline{\Omega}, \tag{24}$$

is a barrier at z. Indeed, φ is harmonic in $\mathbb{R}^n \setminus \{y\}$, $\varphi(z) = 0$, and $\varphi(x) < 0$ for $x \in \mathbb{R}^n \setminus \overline{B_r(y)}$. For $n = 2$, it is again straightforward to check that

$$\varphi(x) = \log \frac{1}{|x-y|} - \log \frac{1}{r}, \qquad x \in \overline{\Omega}, \tag{25}$$

is a barrier at z. □

Remark 9. In fact, we have the following criterion due to Lebesgue: The point $0 \in \partial\Omega$ is regular if any $x \in \Omega$ near 0 satisfies $x_n < f(|x'|)$, where $x' = (x_1, \ldots, x_{n-1})$ and $f(r) = ar^{1/m}$ for some $a > 0$ and $m > 0$. The case $m = 1$ is known as *Zaremba's criterion* or the *exterior cone condition.*

The following example shows that Lebesgue's criterion is nearly optimal in the sense that the criterion would not be valid if $f(r) = a/\log\frac{1}{r}$.

Example 10 (Lebesgue 1913)**.** Let $I = \{(0,0,s) : 0 \leq s \leq 1\} \subset \mathbb{R}^3$ and let

$$v(x) = \int_0^1 \frac{s\,\mathrm{d}s}{|x - p(s)|} \qquad x \in \mathbb{R}^3 \setminus I, \tag{26}$$

where $p(s) = (0,0,s) \in I$. Note that v is the potential produced by a charge distribution on I, whose density linearly varies from 0 to 1. Consequently, we have $\Delta v = 0$ in $\mathbb{R}^3 \setminus I$, and in particular, $v \in C^\infty(\mathbb{R}^3 \setminus I)$. It is easy to compute

$$v(x) = |x - p(1)| - |x| + x_3 \log(1 - x_3 + |x - p(1)|) - x_3 \log(-x_3 + |x|). \tag{27}$$

We will be interested in the behaviour of $v(x)$ as $x \to 0$. First of all, since $-x_3 + |x| \geq 2|x_3|$ for $x_3 \leq 0$, if we send $x \to 0$ while keeping $x_3 \leq 0$, then $v(x) \to 1$. To study what happens when $x_3 > 0$, we write

$$v(x) = v_0(x) - x_3 \log\left(|x_1|^2 + |x_2|^2\right), \tag{28}$$

with

$$v_0(x) = |x - p(1)| - |x| + x_3 \log(1 - x_3 + |x - p(1)|) + x_3 \log(x_3 + |x|). \tag{29}$$

The function v_0 is continuous in $\mathbb{R}^3 \setminus \{0, p(1)\}$ with $v_0(x) \to 1$ as $x \to 0$. Moreover, if we send $x \to 0$ in the region $|x_1|^2 + |x_2|^2 \geq |x_3|^n$ with some n, then we still have $v(x) \to 1$. On the other hand, if we send $x \to 0$ along a curve with $|x_1|^2 + |x_2|^2 = e^{-\alpha/x_3}$ for some constant $\alpha > 0$, then we have $v(x) \to 1 + \alpha$. We also note that because of the singularity at

$x_1 = x_2 = 0$ of the last term in (28), we see that $v(x) \to +\infty$ as x approaches $I \setminus \{0\}$. Now we define $\Omega = \{x : v(x) < 1+\alpha\} \cap B_1$ with a sufficiently large $\alpha > 0$. Then although $v(0)$ can be defined so that v is continuous on $\partial\Omega$, it is not possible to extend v to a function in $C(\overline{\Omega})$.

Next, consider the Dirichlet problem $\Delta u = 0$ in Ω, and $u = v$ on $\partial\Omega$. Let $M = \|u-v\|_{L^\infty(\Omega)}$, and for $\varepsilon > 0$, let $\Omega_\varepsilon = \Omega \setminus \overline{B}_\varepsilon$. Then the function

$$w(x) = \frac{M\varepsilon}{|x|} \pm (u(x) - v(x)), \tag{30}$$

satisfies $\Delta w = 0$ in Ω_ε and $w \geq 0$ on $\partial\Omega_\varepsilon$. By the minimum principle, we have $w \geq 0$ in Ω_ε, which means that

$$|u(x) - v(x)| \leq \frac{M\varepsilon}{|x|}, \qquad x \in \Omega_\varepsilon. \tag{31}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $u = v$. ⊘

If p is an isolated boundary point, i.e., if p is the only boundary point in a neighbourhood of it, then as Zaremba observed, one cannot specify a boundary condition at p because it would be a removable singularity for the harmonic function in the domain. It is shown by Osgood in 1900 that this is the only possible way for a boundary point of a *two dimensional* domain to be irregular.

Theorem 11 (Osgood 1900)**.** *Let $\Omega \subset \mathbb{R}^2$ be open and let $p \in \partial\Omega$ be contained in a component of $\mathbb{R}^2 \setminus \Omega$ which has more than one point (including p). Then p is regular.*

Proof. It will be convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and without loss of generality, to assume that $p = 0$. Let $w \in \mathbb{C}$ be another point so that both p and w are contained in the same connected component of $\mathbb{C} \setminus \Omega$. After a possible scaling, we can assume that $|w| > 1$. Moreover, since regularity is a local property, we can restrict attention to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, that is, we assume that $\Omega \subset \mathbb{D}$. Let $z_0 \in \Omega$ and consider a branch of logarithm near z_0. This branch can be extended to Ω as a single-valued function, for if it were not, there must exist a closed curve in Ω that goes around the origin. However, it is impossible because there is a connected component of $\mathbb{C} \setminus \Omega$ that contains 0 and w. Denoting the constructed branch by log, we claim that $\varphi(z) = \mathrm{Re}(\log z)^{-1}$ is a barrier. Since $\log z$ is a holomorphic function that vanishes nowhere in Ω, we have $\Delta\varphi = 0$ in Ω. Moreover, we have $\varphi(z) \to 0$ as $z \to 0$ and $\varphi < 0$ in Ω because $\mathrm{Re}(\log z) = \log|z|$ and $|z| < 1$ for $z \in \Omega$. This shows that φ is indeed a barrier at 0. □

4. The Dirichlet energy

In this section, we will be introduced to the problem of minimizing the *Dirichlet energy*

$$E(u) = \int_\Omega |\nabla u|^2, \tag{32}$$

subject to the boundary condition $u|_{\partial\Omega} = g$. Recall from the introduction that this approach to the Dirichlet problem was originally suggested around 1847 by William Thomson and Gustav Lejeune-Dirichlet. We start with some simple observations.

Lemma 12. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C^2(\Omega)$ satisfy $E(u) < \infty$.*

- *If $\Delta u = 0$ in Ω, then $E(u+v) > E(u)$ for all nontrivial $v \in \mathscr{D}(\Omega)$.*
- *Conversely, if $E(u+v) \geq E(u)$ for all $v \in \mathscr{D}(\Omega)$, then $\Delta u = 0$ in Ω.*

Proof. Let $v \in \mathscr{D}(\Omega)$ and let $\varepsilon \in \mathbb{R}$. Then we have

$$E(u+\varepsilon v) = E(u) + 2\varepsilon \int_\Omega \nabla u \cdot \nabla v + \varepsilon^2 E(v) = E(u) - 2\varepsilon \int_\Omega v\Delta u + \varepsilon^2 E(v), \tag{33}$$

by Green's first identity and the fact that supp v is compact. The first assertion of the lemma follows by putting $\Delta u = 0$ and $\varepsilon = 1$. For the second assertion, note that

$$2\varepsilon \int_\Omega v\Delta u = E(u) - E(u+\varepsilon v) + \varepsilon^2 E(v) \leq \varepsilon^2 E(v), \tag{34}$$

for all $\varepsilon \in \mathbb{R}$, implying that

$$2\Big|\int_\Omega v\Delta u\Big| \leq |\varepsilon| E(v), \qquad \text{and so} \qquad \int_\Omega v\Delta u = 0. \tag{35}$$

Since v is arbitrary and Δu is continuous, we infer that $\Delta u = 0$ in Ω. □

The second assertion of the preceding lemma tells us that in order to establish existence of a solution to the Dirichlet problem (13), it suffices to show that E has a minimizer in

$$\mathscr{A}_0 = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) : u|_{\partial\Omega} = g\}. \tag{36}$$

In order to obtain a minimizer, one would start with a sequence $\{u_k\} \subset \mathscr{A}_0$ satisfying

$$E(u_k) \to \mu := \inf_{v\in\mathscr{A}_0} E(v) \qquad \text{as} \quad k \to \infty, \tag{37}$$

and then try to show that this sequence (or some subsequence of it) converges to an element $u \in \mathscr{A}_0$ with $E(u) = \mu$. Such a sequence is called a *minimizing sequence*. The difficulty with this plan is that although one can easily establish the existence of some function u such that $u_k \to u$ in a certain sense, the topology in which the convergence $u_k \to u$ occurs is so weak that we cannot imply the membership $u \in \mathscr{A}_0$ from the convergence alone. Initially, e.g., in the works of David Hilbert and Richard Courant, this difficulty was overcame by modifying the sequence $\{u_k\}$ without loosing its minimizing property, so as to be able to say more about the properties of the limit u. However, it was later realized that the following modular approach is more natural and often better suited for generalization.

- First, we show that E has a minimizer in a class that contains $\mathscr{A}_0$ as a subset.
- Then we show that the minimizer we obtained is in fact in $\mathscr{A}_0$.

The division of labor described here, that separates *existence* questions from *regularity* questions, has became the basic philosophy of calculus of variations. Already in 1900, Hilbert proposed existence and regularity questions (for minimization of more general energies) as two individual problems in his famous list.

In a few sections that follow, we will carry out this program for the Dirichlet energy. Before setting up the problem, let us look at a counterexample due to Jacques Hadamard, which is a variation of Friedrich Prym's example from 1871.

Example 13 (Hadamard 1906)**.** Let $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$, and let $u : \mathbb{D} \to \mathbb{R}$ be given in polar coordinates by

$$u(r,\theta) = \sum_{n=1}^{\infty} n^{-2} r^{n!} \sin(n!\theta). \tag{38}$$

It is easy to check that each term of the series is harmonic, and the series converges absolutely uniformly in $\overline{\mathbb{D}}$. Hence u is harmonic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. On the other hand, we have

$$E(u) = \int_{\mathbb{D}} |\nabla u|^2 \geq \int_0^{2\pi}\int_0^{\rho} |\partial_r u(r,\theta)|^2 r \,\mathrm{d}r\,\mathrm{d}\theta = \sum_{n=1}^{\infty} \frac{\pi n!}{2n^4}\rho^{2n!} \geq \sum_{n=1}^{m} \frac{\pi n!}{2n^4}\rho^{2n!}, \tag{39}$$

for any $\rho < 1$ and any integer m. This implies that $E(u) = \infty$. To conclude, there exists a Dirichlet datum $g \in C(\partial\mathbb{D})$ for which the Dirichlet problem is perfectly solvable, but the solution cannot be obtained by minimizing the Dirichlet energy. There is no full equivalence between the Dirichlet problem and the minimization problem.

We are now ready to enter the full mathematical set up of the problem. With $\Omega \subset \mathbb{R}^n$ a domain (not necessarily bounded) and $g \in C(\partial\Omega)$, we would like to minimize E over a class of functions u satisfying the boundary condition $u|_{\partial\Omega} = g$. However, in view of Hadamard's example, we want to make sure that there is at least one function u such that $u|_{\partial\Omega} = g$ and that $E(u) < \infty$. We will implement it by assuming from the beginning that $g \in C^1(\Omega) \cap C(\overline{\Omega})$ and that $E(g) < \infty$, so that the boundary condition now takes the form $u|_{\partial\Omega} = g|_{\partial\Omega}$. Furthermore, we want to give some flexibility in the way we impose boundary conditions. To be specific, we want to include the possibility to require not only the values of $u - g$ vanish at $\partial\Omega$, but also some or all of the derivatives of $u - g$ vanish at $\partial\Omega$. As a device to enforce boundary conditions, we will fix once and for all a linear space $\mathscr{X}$ satisfying the inclusions

$$\mathscr{D}(\Omega) \subset \mathscr{X} \subset C_0^1(\Omega) \equiv \{u \in C^1(\Omega) \cap C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}. \tag{40}$$

and we let

$$\mathscr{A} = \{u \in C^1(\Omega) \cap C(\overline{\Omega}) : u - g \in \mathscr{X}\}. \tag{41}$$

Therefore, the choice $\mathscr{X} = C_0^1(\Omega)$ corresponds to imposing $u|_{\partial\Omega} = g|_{\partial\Omega}$, whereas the choice $\mathscr{X} = \mathscr{D}(\Omega)$ corresponds to requiring all derivatives of $u - g$ vanish at $\partial\Omega$. The textbook way of doing this would be to simply take $\mathscr{X} = \mathscr{D}(\Omega)$ from the beginning, which may appear a bit strange as it is not *a priori* clear why we need to impose conditions on the derivatives of $u - g$ near $\partial\Omega$. However, it turns out that any choice of $\mathscr{X}$, so long as $\mathscr{D}(\Omega) \subset \mathscr{X} \subset C_0^1(\Omega)$, would lead to the same outcome. Since we will not prove this equivalence in these notes, to be transparent, we want to keep the distinction between different choices of $\mathscr{X}$ at least for a while. We remark that the differentiability condition is now C^1 in (41) as opposed to C^2 in (36), because the Dirichlet energy makes perfect sense for C^1 functions. This is not very relevant, because as we shall see, we will eventually be using an even larger class functions.

The next step is to consider a *minimizing sequence*. Let

$$\mu = \inf_{v \in \mathscr{A}} E(v). \tag{42}$$

It is obvious that $0 \leq \mu < \infty$, because $g \in \mathscr{A}$ and $E(g) < \infty$. By definition of infimum, there exists a sequence $\{u_k\} \subset \mathscr{A}$ satisfying

$$E(u_k) \to \mu \qquad \text{as} \quad k \to \infty. \tag{43}$$

We will see that there is a natural topology associated to the energy E in which we have the convergence $u_k \to u$ to some function u. However, this topology cannot be very strong, as the following example illustrates.

Example 14 (Courant)**.** Consider the Dirichlet problem on the unit disk $\mathbb{D} \subset \mathbb{R}^2$ with the homogeneous Dirichlet boundary condition. The solution is $u \equiv 0$, which also minimizes the Dirichlet energy. For $k \in \mathbb{N}$, let

$$u_k(r,\theta) = \begin{cases} ka_k & \text{for } r < e^{-2k}, \\ -a_k(k + \log r) & \text{for } e^{-2k} < r < e^{-k}, \\ 0 & \text{for } e^{-k} < r < 1, \end{cases} \tag{44}$$

given in polar coordinates. These are continuous, piecewise smooth functions with

$$E(u_k) = 2\pi \int_0^1 |\partial_r u_k|^2 \, r \, \mathrm{d}r = 2\pi a_k^2 \log r \Big|_{e^{-2k}}^{e^{-k}} = 2\pi k a_k^2. \tag{45}$$

Upon choosing $a_k = k^{-2/3}$, we can ensure that $\{u_k\}$ is a minimizing sequence. However, $u_k(0) = ka_k = k^{1/3}$ diverges as $k \to \infty$. In any case, observe that u_k converges to $u \equiv 0$ in some averaged sense.

Exercise 5. In the context of the preceding example, construct a minimizing sequence of piecewise smooth functions satisfying the homogeneous boundary condition, which diverges in a set that is dense in $\mathbb{D}$. Show that this sequence converges to $u \equiv 0$ in L^2.

In order to illustrate the main ideas clearly, before dealing with the Dirichlet energy (32), we would like to consider the problem of minimizing the modified energy

$$E_*(u) = \int_\Omega \left(|\nabla u|^2 + |u|^2\right). \tag{46}$$

The admissible set $\mathscr{A}$ will stay the same, as in (41), and we will assume that

$$\mu_* = \inf_{v\in\mathscr{A}} E_*(v) < \infty. \tag{47}$$

If Ω is bounded, $\mu_* < \infty$ if and only if $\mu < \infty$, because the second term under the integral in (46) is integrable for any $u \in \mathscr{A}$. One can also show that minimizing E_* corresponds to the boundary value problem $\Delta u = u$ in Ω and $u = g$ on $\partial\Omega$.

Exercise 6. Establish an analogue of Lemma 12 for the modified energy E_*. In particular, show that if $u \in C^2(\Omega)$ satisfies $E_*(u + v) \geq E_*(u)$ for all $v \in \mathscr{D}(\Omega)$, then $\Delta u = u$ in Ω.

Pick a minimizing sequence for E_*, i.e., let $\{u_k\} \subset \mathscr{A}$ be such that

$$E_*(u_k) \to \mu_* \qquad \text{as} \quad k \to \infty. \tag{48}$$

Note that $E(u) = \langle u, u\rangle_*$, where $\langle\cdot,\cdot\rangle_*$ is the symmetric bilinear form given by

$$\langle u, v\rangle_* = \int_\Omega \left(\nabla u \cdot \nabla v + uv\right). \tag{49}$$

Any symmetric bilinear form satisfies the *parallelogram law*:

$$\langle u - v, u - v\rangle_* + \langle u + v, u + v\rangle_* = 2\langle u, u\rangle_* + 2\langle v, v\rangle_*, \tag{50}$$

which reveals that

$$E_*(u_j - u_k) = 2E_*(u_j) + 2E_*(u_k) - 4E_*\big(\frac{u_j + u_k}{2}\big) \leq 2E_*(u_j) + 2E_*(u_k) - 4\mu_*, \tag{51}$$

where the inequality is because of the fact that $\frac{u_j+u_k}{2} \in \mathscr{A}$. Since $\{u_j\}$ is a minimizing sequence, we have $E_*(u_j - u_k) \to 0$ as $j, k \to \infty$. We would have said that $\{u_j\}$ is a Cauchy sequence, for instance, if there was some norm of $u_j - u_k$, rather than $E_*(u_j - u_k)$, that is going to 0. As it turns out, this is indeed true: The quantity $\|\cdot\|_{H^1(\Omega)} = \sqrt{E_*}$, that is, $\|\cdot\|_{H^1(\Omega)}$ given by

$$\|u\|^2_{H^1(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} = \int_\Omega \left(|\nabla u|^2 + |u|^2\right), \tag{52}$$

is a norm for functions in

$$\tilde{C}^1(\Omega) = \{u \in C^1(\Omega) : \|u\|_{H^1(\Omega)} < \infty\}. \tag{53}$$

The space $\tilde{C}^1(\Omega)$ is a proper subset of $C^1(\Omega)$, for instance, by Hadamard's example, although much simpler examples can be constructed that exploit growth, rather than oscillation, near the boundary of Ω. To conclude, (51) implies that the sequence $\{u_j\}$ is Cauchy with respect to this new norm:

$$\|u_j - u_k\|_{H^1(\Omega)} \to 0, \qquad \text{as} \quad j, k \to \infty. \tag{54}$$

The reader must have recognized why we modified the Dirichlet energy: It is precisely to make $\sqrt{E_*}$ a norm. For the Dirichlet energy, $\sqrt{E}$ is still a norm for functions in $C^1(\Omega)$ satisfying the homogeneous boundary condition, but to show this one needs a bit more machinery, in particular the Friedrichs inequality (Lemma 24 below). Note that since $u_j - u_k$ satisfies the homogeneous boundary condition, this would have been sufficient for us. The Dirichlet energy E will be taken up after the treatment of E_* which is a bit simpler.

5. Strong derivatives and weak solutions

Returning back to minimizing E_*, we have shown that any minimizing sequence is a Cauchy sequence with respect to the norm $\|\cdot\|_{H^1(\Omega)}$. Now, if $\tilde{C}^1(\Omega)$ was complete with respect to the norm $\|\cdot\|_{H^1(\Omega)}$, there would have been $u \in \tilde{C}^1(\Omega)$ such that $\|u_j - u\|_{H^1(\Omega)} \to 0$. However, it is a fact of life that $\tilde{C}^1(\Omega)$ is *not complete* with respect to the norm $\|\cdot\|_{H^1(\Omega)}$.

Exercise 7. Let $v(x) = \log\log(2/|x|)$ and let $v_k \in C(\mathbb{D})$ be defined by $v_k(x) = \min\{k, v(x)\}$. Show that the norms $\|v_k\|_{H^1(\mathbb{D})}$ are uniformly bounded. Exhibit a sequence $\{u_k\} \subset \tilde{C}^1(\mathbb{D})$ that is Cauchy with respect to the norm $\|\cdot\|_{H^1(\mathbb{D})}$, whose limit is not essentially bounded.

If we ignore every aspect of the problem except the fact that $\{u_j\}$ is a Cauchy sequence, the best thing we can do is to consider the *completion* of $\tilde{C}^1(\Omega)$. What we get in this way is a member of a large family of function spaces called *Sobolev spaces*, named in honour of Sergei Sobolev, who initiated the systematic study of these spaces.

Definition 15. We define the *Sobolev space* $H^1(\Omega)$ as the completion of $\tilde{C}^1(\Omega)$ with respect to the norm $\|\cdot\|_{H^1(\Omega)}$. We also define $H^1\mathscr{X}$ as the closure of $\mathscr{X} \cap H^1(\Omega)$ in $H^1(\Omega)$. The space $H^1\mathscr{D}(\Omega)$ is denoted by $H^1_0(\Omega)$.

If there is no risk of confusion, we will simply write $\|\cdot\|_{H^1}$ omitting from the notation the domain Ω, and call this norm the H^1-*norm*. By construction, $H^1\mathscr{X}$ is a closed subspace of $H^1(\Omega)$. For now, $H^1(\Omega)$ is a space whose elements are equivalence classes of Cauchy sequences. We want to identify $H^1(\Omega)$ with a subspace of $L^2(\Omega)$, which would give us a concrete handle on $H^1(\Omega)$. We start with the following observation: If a sequence $\{\phi_k\} \subset \tilde{C}^1(\Omega)$ is Cauchy with respect to the H^1-norm, then each of the sequences $\{\phi_k\}$ and $\{\partial_i\phi_k\}$, where $i = 1, \ldots, n$, is Cauchy in $L^2(\Omega)$. In particular, since $L^2(\Omega)$ is a complete space, there exists a function $u \in L^2(\Omega)$ such that $\phi_k \to u$ in $L^2(\Omega)$ as $k \to \infty$. This defines a map from $H^1(\Omega)$ into $L^2(\Omega)$: It sends the element of $H^1(\Omega)$ represented by the sequence $\{\phi_k\}$ to $u \in L^2(\Omega)$. Let us call this map $J_0 : H^1(\Omega) \to L^2(\Omega)$. We will eventually prove that J_0 is injective, identifying $H^1(\Omega)$ as a subspace of $L^2(\Omega)$. For the time being, let us look into the range of J_0. It is clear that a function $u \in L^2(\Omega)$ is in the range of J_0 if and only if there exist a sequence $\{\phi_k\} \subset \tilde{C}^1(\Omega)$ and functions $v_i \in L^2(\Omega)$ for $i = 1, \ldots, n$, such that

$$\phi_k \to u, \qquad \text{and} \qquad \partial_i\phi_k \to v_i, \quad (i = 1, \ldots, n), \tag{55}$$

with all convergences taking place in $L^2(\Omega)$. This leads to the concept of strong derivatives, which is based on *approximation*.

Definition 16. For $u, v \in L^2_{\text{loc}}(\Omega)$, we say that $v = \partial_i u$ in the *strong* L^2-*sense*, or that v is a *strong* L^2 *derivative* of u, if for each compact set $K \subset \Omega$, there exists a sequence $\{\phi_k\} \subset C^1(K)$ such that

$$\phi_k \to u \qquad \text{and} \qquad \partial_i\phi_k \to v \qquad \text{as} \quad k \to \infty, \tag{56}$$

with both convergences taking place in $L^2(K)$.

In particular, if $u \in L^2(\Omega)$ is in the range of J_0, then u is strongly L^2 differentiable.

Example 17. a) Let $u \in C^1(\Omega)$. Then taking the constant sequence $\phi_k = u$ (for all k) shows that the classical derivative $\partial_i u$ is also a strong L^2 derivative of u.

b) Let $u(x) = |x|$ for $x \in \mathbb{R}$, and let $\phi_k(x) = \sqrt{x^2+\varepsilon^2}$ with $\varepsilon = 1/k$. Obviously, we have $\phi_k \in C^\infty(\mathbb{R})$. From the Maclaurin series of $\sqrt{1+x}$, we get

$$\sqrt{x^2+\varepsilon^2} = |x|\sqrt{1+\frac{\varepsilon^2}{x^2}} = |x|(1+e(x)), \qquad |e(x)| \leq \frac{\varepsilon^2}{x^2}, \tag{57}$$

for $\varepsilon < |x|$, and hence

$$\big|\phi_k(x) - |x|\big| = \big|\sqrt{x^2+\varepsilon^2} - |x|\big| \leq \frac{\varepsilon^2}{|x|} \qquad \text{for} \quad |x| > \varepsilon. \tag{58}$$

On the other hand, we have $\phi_k(x) \leq \sqrt{2}\varepsilon$ for $|x| \leq \varepsilon$, so that

$$\int_{-\varepsilon}^{\varepsilon} \big|\phi_k(x) - |x|\big|^2 \,\mathrm{d}x \leq (4\varepsilon^2 + 2\varepsilon^2)\cdot 2\varepsilon. \tag{59}$$

Together with (58) this implies that $\phi_k \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R})$, because

$$\int_{-a}^{a} \big|\phi_k(x) - |x|\big|^2 \,\mathrm{d}x \leq 12\varepsilon^3 + 2a\varepsilon, \tag{60}$$

for any $a > 0$. Now we look at

$$\phi_k'(x) = \frac{x}{\sqrt{x^2+\varepsilon^2}}, \tag{61}$$

and would like to show that ϕ_k' converges in $L^2_{\mathrm{loc}}(\mathbb{R})$ to the sign function

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0. \end{cases} \tag{62}$$

This would show that the sign function is a derivative of the absolute value function in the strong L^2-sense. Since ϕ_k' and sgn are both odd functions, it suffices to consider only the half line $x > 0$. We observe that $\phi_k'(x) \leq 1$, and that

$$1 - \phi_k'(x) = \frac{\sqrt{x^2+\varepsilon^2} - x}{\sqrt{x^2+\varepsilon^2}} \leq \frac{\varepsilon^2}{x^2}, \qquad \text{for} \quad |x| > \varepsilon, \tag{63}$$

where we have used (58). Then for any fixed $a > 0$ and k large (hence ε small), we compute

$$\int_0^a |\phi_k'(x) - 1|^2 \,\mathrm{d}x = \int_0^{\sqrt{\varepsilon}} |\phi_k'(x) - 1|^2 \,\mathrm{d}x + \int_{\sqrt{\varepsilon}}^a |\phi_k'(x) - 1|^2 \,\mathrm{d}x \leq 4\sqrt{\varepsilon} + a\varepsilon, \tag{64}$$

which confirms the desired convergence.

Proceeding further, for each $i \in \{1, \ldots, n\}$, we can define the map $J_i : H^1(\Omega) \to L^2(\Omega)$ that captures the L^2-limit of $\partial_i\phi_k$ as $k \to \infty$, where $\{\phi_k\}$ is a sequence representing an element of $H^1(\Omega)$. The composite map $J = (J_0, \ldots, J_n) : H^1(\Omega) \to L^2(\Omega)^{n+1}$ is clearly injective, because if $\lim \phi_k = \lim \psi_k$ and $\lim \partial_i\phi_k = \lim \partial_i\psi_k$ for all i, then the mixed sequence $\phi_1, \psi_1, \phi_2, \psi_2, \ldots$ is Cauchy in $H^1(\Omega)$, and so the two sequences $\{\phi_k\}$ and $\{\psi_k\}$ represent the same element of $H^1(\Omega)$. We see that the injectivity of J_0 would follow once we have shown that for any $U \in H^1(\Omega)$, the components $J_1U, \ldots, J_nU$ are uniquely determined by J_0U alone. The following result confirms this.

Lemma 18. *Strong derivatives are unique if they exist. Moreover, if $u \in L^2_{\mathrm{loc}}(\Omega)$ is strongly L^2 differentiable, then we have the integration by parts formula*

$$\int_\Omega \varphi\partial_i u = -\int_\Omega u\partial_i\varphi, \qquad \varphi \in C^1_c(\Omega). \tag{65}$$

Proof. Suppose that both $v, w \in L^2_{\text{loc}}(\Omega)$ are strong L^2 derivative of $u \in L^2_{\text{loc}}(\Omega)$. We want to show that $v = w$ almost everywhere. Let $\varphi \in C^1_c(\Omega)$, and put $K = \operatorname{supp} \varphi$. Then there is a sequence $\{v_k\} \subset C^1(K)$ such that $v_k \to u$ and $\partial_i v_k \to v$ as $k \to \infty$, both convergences in $L^2(K)$. From the usual integration by parts, we have

$$\int_\Omega v\varphi = \int_\Omega (v - \partial_i v_k)\varphi + \int_\Omega \varphi \partial_i v_k = \int_\Omega (v - \partial_i v_k)\varphi - \int_\Omega v_k \partial_i \varphi, \tag{66}$$

hence

$$\begin{aligned} \Big| \int_\Omega v\varphi + \int_\Omega u\partial_i\varphi \Big| &\le \int_\Omega |v - \partial_i v_k||\varphi| + \int_\Omega |u - v_k||\partial_i \varphi| \\ &\le \|v - \partial_i v_k\|_{L^2(K)} \|\varphi\|_{L^2} + \|u - v_k\|_{L^2(K)} \|\partial_i \varphi\|_{L^2}, \end{aligned} \tag{67}$$

showing that the formula (65) is valid. The same reasoning applies to w, which means that

$$\int_\Omega (v - w)\varphi = \int_\Omega (u - u)\partial_i \varphi = 0. \tag{68}$$

Since $\varphi \in C^1_c(\Omega)$ is arbitrary, by the du Bois-Reymond lemma (Lemma 27 in §7) we conclude that $v = w$ almost everywhere. □

Remark 19. The heart of the uniqueness argument was the integration by parts formula (65). We will see in §8 that in fact the property (65) characterizes strong derivatives.

In terms of the new concepts we have just defined, we can say that the minimizing sequence $\{u_j\} \subset \mathscr{A}$ converges to some $u \in H^1(\Omega)$. Moreover, from the definition (41), the sequence $\{v_k\}$ defined by $v_k = u_k - g$ is in $\mathscr{X}$, and it is Cauchy in $H^1(\Omega)$, hence $u - g \in H^1\mathscr{X}$. We emphasize here that the only part of the boundary condition that survives the limit process is $u - g \in H^1\mathscr{X}$, and this must be understood as a generalized form of the Dirichlet boundary condition. The energy E_* is a continuous function on $\tilde{C}^1(\Omega)$ with respect to the H^1-norm, that can be seen, for instance, from the inequality

$$|E_*(\phi) - E_*(\psi)| \le \|\phi + \psi\|_{H^1(\Omega)} \|\phi - \psi\|_{H^1(\Omega)}, \qquad \phi, \psi \in \tilde{C}^1(\Omega). \tag{69}$$

Hence E_* can be extended to a continuous function on $H^1(\Omega)$ in a unique way. Keeping the notation E_* for this extension, we have

$$E_*(u) = E_*(\lim u_j) = \lim E_*(u_j) = \mu_*. \tag{70}$$

We cannot say that u minimizes the energy E_* over $\mathscr{A}$, because we have not ruled out the possibility $u \notin \mathscr{A}$. What we *can* say though is that u minimizes E_* over the set

$$\tilde{\mathscr{A}} = \{g + v : v \in H^1\mathscr{X}\} \supset \mathscr{A}, \tag{71}$$

since $u - g \in H^1\mathscr{X}$ and for any $w \in \tilde{\mathscr{A}}$ there is a sequence $\{w_k\} \in \mathscr{A}$ converging to w in $H^1(\Omega)$, meaning that

$$E_*(w) = E_*(\lim w_k) = \lim E_*(w_k) \ge \mu_*. \tag{72}$$

Let us now try to derive a differential equation from the minimality of u, as was done in Lemma 12. It is easy to see that $E_*(w)$ can be calculated by the same formula

$$E_*(w) = \int_\Omega \left(|\nabla w|^2 + |w|^2 \right), \tag{73}$$

also for $w \in H^1(\Omega)$, with $\nabla w = (\partial_1 w, \ldots, \partial_n w)$ understood in the strong L^2-sense. In light of this, we have

$$E_*(u) \le E_*(u + \varepsilon v) = E_*(u) + \varepsilon^2 E_*(v) + 2\varepsilon \int_\Omega \left(\nabla u \cdot \nabla v + uv \right), \tag{74}$$

for $\varepsilon \in \mathbb{R}$ and $v \in H^1\mathscr{X}$, which then implies that

$$\langle u, v\rangle_{H^1} := \int_\Omega (\nabla u \cdot \nabla v + uv) = 0, \qquad \text{for all} \quad v \in H^1\mathscr{X}. \tag{75}$$

We cannot go any further because we cannot quite move the derivatives from v to u in such a low regularity setting (cf. Exercise 6). Until we can prove that u is indeed smooth, we will have to work with (75) as it is.

Definition 20. If $u \in H^1(\Omega)$ satisfies (75), then we say that u solves $\Delta u = u$ in Ω in the *weak sense*, or that u is a *weak solution* of $\Delta u = u$ in Ω. In the same spirit, we call (75) the *weak formulation* of the equation $\Delta u = u$ in Ω.

We have practically proved the following result.

Theorem 21. *Let $g \in H^1(\Omega)$. Then there exists a unique $u \in H^1(\Omega)$ satisfying (75) and $u - g \in H^1\mathscr{X}$. In other words, there is a unique weak solution of $\Delta u = u$ with $u - g \in H^1\mathscr{X}$.*

Proof. For uniqueness, let us start as usual by assuming that there exist two such functions $u_1, u_2 \in H^1(\Omega)$. Then $w = u_1 - u_2 \in H^1\mathscr{X}$, and by linearity, we have

$$\int_\Omega (\nabla w \cdot \nabla v + wv) = 0, \tag{76}$$

for all $v \in H^1\mathscr{X}$. Taking $v = w$ gives $\|w\|_{H^1} = 0$, hence $w = 0$.

Existence had already been established, modulo the fact that we now allow $g \in H^1(\Omega)$. For completeness, let us sketch a proof. We define the admissible set $\tilde{\mathscr{A}}$ as in (71), and take a minimizing sequence $\{u_j\} \subset \tilde{\mathscr{A}}$, that is, a sequence satisfying

$$E_*(u_j) \to \mu_* = \inf_{v \in \tilde{\mathscr{A}}} E_*(v). \tag{77}$$

We have $0 \leq \mu_* < \infty$, since $E_*(g) = \|g\|_{H^1}^2 < \infty$. The argument (51) shows that $\{u_j\}$ is Cauchy in $H^1(\Omega)$, and hence there is $u \in H^1(\Omega)$ such that $u_j \to u$ in H^1. By continuity of E_*, i.e., the argument (70), we have $E_*(u) = \mu_*$. Moreover, the sequence $\{u_j - g\}$ is Cauchy in $H^1(\Omega)$, and $H^1\mathscr{X}$ is a closed subspace of $H^1(\Omega)$, implying that $u - g \in H^1\mathscr{X}$. Finally, the argument (74) confirms that (75) is satisfied. □

Exercise 8 (Stability). Let $u_1 \in H^1(\Omega)$ and $u_2 \in H^1(\Omega)$ be the weak solutions of $\Delta u = u$ satisfying $u_1 - g_1 \in H_0^1(\Omega)$ and $u_2 - g_2 \in H_0^1(\Omega)$, where $g_1, g_2 \in H^1(\Omega)$. Show that

$$\|u_1 - u_2\|_{H^1} \leq \|g_1 - g_2\|_{H^1}. \tag{78}$$

6. Minimization of the Dirichlet energy

We have proved that the energy E_* attains its minimum over the set $\tilde{\mathscr{A}}$, and that the minimizer is the weak solution to $\Delta u = u$ in Ω. If we can show that u is smooth, then this would imply that $\Delta u = u$ pointwise in Ω. Leaving the smoothness question aside for the moment, now we would like to return to our original goal, that is to minimize the Dirichlet energy E over $\tilde{\mathscr{A}}$. To this end, let us try to imitate and adapt the proof of Theorem 21. Recall that the admissible set $\tilde{\mathscr{A}}$ is defined in (71) with some $g \in H^1(\Omega)$. Let $\{u_j\} \subset \tilde{\mathscr{A}}$ be a minimizing sequence, i.e., let

$$E(u_j) \to \mu = \inf_{v \in \tilde{\mathscr{A}}} E(v). \tag{79}$$

We have $0 \leq \mu < \infty$, since $E(g) = \|\nabla g\|_{L^2}^2 < \infty$. Repeating the argument (51), we find that

$$\|\nabla(u_j - u_k)\|_{L^2} \to 0, \qquad \text{as} \quad j, k \to \infty. \tag{80}$$

As $\|\nabla \cdot\|_{L^2}$ is only a part of the H^1-norm, we cannot directly say that the sequence $\{u_j\}$ is Cauchy in H^1. In particular, the fact that $\|\nabla v\|_{L^2} = 0$ would only mean that v is a constant function. However, if we know that $v = 0$ on $\partial\Omega$, then this constant must be 0. This is the intuitive reason behind the *Friedrichs inequality*[1]

$$\|v\|_{H^1} \leq c\|\nabla v\|_{L^2}, \qquad v \in H^1\mathscr{X}, \tag{81}$$

where $c > 0$ is a constant. Under the assumption that the Friedrichs inequality is true, from (80) it is immediate that $\{u_j\}$ is Cauchy in H^1, because $u_j - u_k \in H^1\mathscr{X}$. Proceeding as in the proof of Theorem 21, we conclude that $u_j \to u$ in H^1 for some $u \in H^1(\Omega)$ satisfying $E(u) = \mu$ and $u - g \in H^1\mathscr{X}$.

Exercise 9. Show that the function u from the preceding paragraph satisfies

$$\int_\Omega \nabla u \cdot \nabla v = 0, \qquad \text{for all} \quad v \in H^1\mathscr{X}. \tag{82}$$

Show also that there is a unique $u \in H^1(\Omega)$ satisfying (82) and $u - g \in H^1\mathscr{X}$. $\oslash$

Definition 22. If $u \in H^1(\Omega)$ satisfies (82), then we say that u solves $\Delta u = 0$ in Ω in the *weak sense*, or that u is a *weak solution* of $\Delta u = 0$ in Ω. We call (82) the *weak formulation* of the equation $\Delta u = 0$ in Ω.

Modulo the proof of Friedrichs' inequality that will follow, we have established the following.

Theorem 23. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $g \in H^1(\Omega)$. Then there exists a unique $u \in H^1(\Omega)$ satisfying $u - g \in H^1\mathscr{X}$ and (82). In other words, there is a unique weak solution of $\Delta u = 0$ with $u - g \in H^1\mathscr{X}$.*

Now we prove the Friedrichs inequality.

Lemma 24 (Friedrichs inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then we have*

$$\|v\|_{L^2} \leq \operatorname{diam}(\Omega)\|\nabla v\|_{L^2}, \qquad \textit{for all} \quad v \in H^1C^1_0(\Omega). \tag{83}$$

Proof. First, we will prove the inequality for $v \in C^1_0(\Omega)$. Let us extend v by 0 outside Ω so that we have $v \in C(\mathbb{R}^n)$. This function is C^1 except possibly at $\partial\Omega$. Without loss of generality, assume that $\Omega \subset (0, a)^n$ for some $a > 0$. Then for $x \in \Omega$, we have

$$|v(x)| = \Big|\int_0^{x_n} \partial_n v(x', t)\,\mathrm{d}t\Big| \leq \int_0^a |\partial_n v(x', t)|\,\mathrm{d}t, \tag{84}$$

where $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. The function $t \mapsto |\partial_n v(x', t)|$ is Riemann integrable because it has at most countable many discontinuities. Now using the Cauchy-Bunyakowsky-Schwarz inequality and squaring, we get

$$|v(x)|^2 \leq a \int_0^a |\partial_n v(x', t)|^2\,\mathrm{d}t, \tag{85}$$

which, upon integrating along $x' = \text{const}$, gives

$$\int_0^a |v(x', t)|^2\,\mathrm{d}t \leq a^2 \int_0^a |\partial_n v(x', t)|^2\,\mathrm{d}t. \tag{86}$$

Then we integrate over $x' \in (0, a)^{n-1}$, and obtain

$$\int_\Omega |v|^2 = \int_{(0,d)^n} |v|^2 \leq a^2 \int_{(0,a)^n} |\partial_n v|^2 = a^2 \int_\Omega |\partial_n v|^2 \leq a^2 \int_\Omega |\nabla v|^2. \tag{87}$$

This establishes the inequality for $v \in C^1_0(\Omega)$.

[1] It is sometimes called the *Poincaré inequality*, although the latter term is used more commonly to refer to the same inequality for functions $v \in H^1(\Omega)$ with vanishing mean.

Now let $v \in H^1_0(\Omega)$. Then by definition there exists a sequence $\{v_k\} \subset C^1_0(\Omega)$ such that $v_k \to v$ in $H^1(\Omega)$. The triangle inequality gives

$$\begin{aligned}\|v\|_{L^2} &\leq \|v_k\|_{L^2} + \|v - v_k\|_{L^2} \leq a\|\nabla v_k\|_{L^2} + \|v - v_k\|_{L^2} \\ &\leq a\|\nabla v\|_{L^2} + a\|\nabla v_k - \nabla v\|_{L^2} + \|v - v_k\|_{L^2},\end{aligned} \tag{88}$$

and since the last two terms can be made arbitrarily small, the lemma follows. □

7. Interior regularity: Weyl's lemma

Now that we have established the existence of a minimizer for the Dirichlet energy, in this section, we want to look at how smooth the minimizer is, and if the minimizer satisfies the equation $\Delta u = 0$ in the classical sense. Both questions can be answered simultaneously and affirmatively, as was done by Hermann Weyl in 1940.

Our approach will be to construct a sequence $\{u_j\}$ of harmonic functions such that $u_j \to u$ in L^1_{loc}, which would then establish the desired result since harmonic functions are closed under the convergence in L^1_{loc}. To construct such an approximating sequence, we will employ the technique of *mollifiers* due to Jean Leray, Sergei Sobolev and Kurt Otto Friedrichs, as it is also useful in many other problems. Let $\rho \in \mathscr{D}(B_1)$ where $B_1 \subset \mathbb{R}^n$ is the unit ball, satisfying $\rho \geq 0$ and $\int \rho = 1$. Then we define $\rho_\varepsilon \in \mathscr{D}(B_\varepsilon)$ for $\varepsilon > 0$ by

$$\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon). \tag{89}$$

It is easy to see that $\int \rho_\varepsilon = 1$. Given $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open, let

$$u_\varepsilon(x) = \int_\Omega \rho_\varepsilon(x - y)u(y)\,\mathrm{d}y, \qquad x \in \Omega. \tag{90}$$

Note that for each $x \in \Omega$, the integral defining $u_\varepsilon(x)$ makes sense for all sufficiently small $\varepsilon > 0$. The function u_ε could be called a mollified version of u, because it is the outcome of a local averaging process, and as we shall see, u_ε is a smooth function.

Theorem 25. *In this setting, we have the followings.*
a) If $u \in C(\Omega)$, then $u_\varepsilon \to u$ locally uniformly as $\varepsilon \to 0$.
b) If $u \in L^q_{\mathrm{loc}}(\Omega)$ for some $1 \leq q < \infty$, then $u_\varepsilon \to u$ in $L^q_{\mathrm{loc}}(\Omega)$ as $\varepsilon \to 0$.

Proof. a) Making use of the facts $\int \rho_\varepsilon = 1$ and $\rho_\varepsilon \geq 0$, we can write

$$|u(x) - u_\varepsilon(x)| \leq \int_\Omega \rho_\varepsilon(x - y)|u(x) - u(y)|\,\mathrm{d}y \leq \sup_{y \in B_\varepsilon(x)} |u(x) - u(y)| = \omega(x, \varepsilon), \tag{91}$$

where the last equality defines the function $\omega : K \times (0, \varepsilon_0) \to \mathbb{R}$, with $K \subset \Omega$ an arbitrary compact set and $\varepsilon_0 > 0$ small, depending on K. Since u is continuous, ω is continuous in $K \times (0, \varepsilon_0)$, and moreover ω can be continuously extended to $K \times [0, \varepsilon_0)$ with $\omega(\cdot, 0) = 0$. This shows that $\omega(x, \varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly in $x \in K$, and part a) follows.

b) Let $K \subset \Omega$ and $K' \subset \Omega$ be compact sets, with K contained in the interior of K'. Then the Hölder inequality gives

$$|u_\varepsilon(x)|^q \leq \big(\int \rho_\varepsilon\big)^{q-1} \int_{K'} \rho_\varepsilon(x - y)|u(y)|^q\,\mathrm{d}y, \tag{92}$$

and integrating over $x \in K$, we get

$$\int_K |u_\varepsilon|^q \leq \int_{K'} \big(\int_K \rho_\varepsilon(x - y)\,\mathrm{d}x\big)|u(y)|^q\,\mathrm{d}y \leq \int_{K'} |u|^q, \tag{93}$$

for small $\varepsilon > 0$. Now let $\delta > 0$ be an arbitrary small number, and let $\phi \in C(K')$ be such that $\|\phi - u\|_{L^q(K')} < \delta$. The existence of such ϕ is guaranteed by the standard density result, which we recall below in Lemma 26. From the bound we just proved, taking into account the linearity of the mollification process, we have

$$\|\phi_\varepsilon - u_\varepsilon\|_{L^q(K)} \leq \|\phi - u\|_{L^q(K')} < \delta. \tag{94}$$

Finally, we use the triangle inequality to obtain

$$\begin{aligned} \|u_\varepsilon - u\|_{L^q(K)} &\leq \|u_\varepsilon - \phi_\varepsilon\|_{L^q(K)} + \|\phi_\varepsilon - \phi\|_{L^q(K)} + \|\phi - u\|_{L^q(K)} \\ &< \|\phi_\varepsilon - \phi\|_{L^q(K)} + 2\delta \\ &\leq \mathrm{vol}(K)^{1/q} \sup_K |\phi_\varepsilon - \phi| + 2\delta, \end{aligned} \tag{95}$$

which, by part a), implies that $\|u_\varepsilon - u\|_{L^q(K)} < 3\delta$ for all sufficiently small ε, and since $\delta > 0$ is arbitrary, the claim follows. □

We now give a proof of the density result we have used.

Lemma 26. *Let $K \subset \mathbb{R}^n$ be a compact set, and let $1 \leq q < \infty$. Then the space of continuous functions on K is dense in $L^q(K)$.*

Proof. Strictly speaking, an element of $L^q(K)$ is an equivalence class of functions that differ on sets of measure zero. We assume that $g : K \to \mathbb{R}$ is a member of such an equivalence class, and shall prove that for any $\varepsilon > 0$, there is $v \in \mathbb{C}(K)$ such that $\|g - v\|_{L^q(K)} < \varepsilon$. This will suffice since for any other member $\tilde{g}$ of the same class, it holds that $\|g - v\|_{L^q(K)} = \|\tilde{g} - v\|_{L^q(K)}$. By decomposing g into its positive and negative parts, we can assume that g takes only nonnegative values, i.e., that $g : K \to [0, \infty)$. Then for $m \in \mathbb{N}$, we define

$$v_m = \sum_{k=0}^{2^{2m}-1} \frac{k}{2^m} \chi_{A_k} + 2^m \chi_B, \tag{96}$$

where $A_k = \{x \in K : \frac{k}{2^m} < g(x) \leq \frac{k+1}{2^m}\}$ and $B = \{x \in K : g(x) > 2^m\}$. For any set S, the characteristic function χ_S is defined as $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise. By construction, the sequence v_m is nondecreasing, and $v_m \to g$ pointwise. Since g is measurable, the sets A_k and B are also measurable, and so are the functions v_m. Moreover, we have

$$|g - v_m|^q \leq 2^{q-1}|g|^q + 2^{q-1}|v_m|^q \leq 2^q |g|^q, \tag{97}$$

which, combined with Lebesgue's dominated convergence theorem, implies that

$$\|g - v_m\|^q_{L^q(K)} = \int_K |g - v_m|^q \to 0, \qquad \text{as} \quad m \to \infty. \tag{98}$$

Thus, the functions of the form (96), that is, the simple functions, are dense in $L^q(K)$. To complete the proof, it suffices to approximate simple functions by continuous functions, or simpler still, approximate the characteristic function of an arbitrary measurable set $A \subset K$ by continuous functions. By regularity of the Lebesgue measure, for any given $\varepsilon > 0$, there exist a compact set $\mathscr{K}$ and an open set $\mathscr{O} \subset \mathbb{R}^n$, such that $\mathscr{K} \subset A \subset \mathscr{O}$ and $|\mathscr{O} \setminus \mathscr{K}| < \varepsilon$, where $|\cdot|$ denotes the Lebesgue measure. Now we define

$$f(x) = \frac{\mathrm{dist}(x, \mathbb{R}^n \setminus \mathscr{O})}{\mathrm{dist}(x, \mathscr{K}) + \mathrm{dist}(x, \mathbb{R}^n \setminus \mathscr{O})}, \qquad x \in \mathbb{R}^n, \tag{99}$$

where $\mathrm{dist}(x, B) = \inf_{y \in B} |x - y|$ for any set $B \subset \mathbb{R}^n$. We have $0 \leq f \leq 1$ everywhere, $f(x) = 1$ for $x \in \mathscr{K}$ and $f(x) = 0$ for $x \in \mathbb{R}^n \setminus \mathscr{O}$. Therefore

$$\|\chi_A - f\|^q_{L^q(K)} \leq \|\chi_A - f\|^q_{L^q(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\chi_A - f|^q \leq |\mathscr{O} \setminus \mathscr{K}| < \varepsilon, \tag{100}$$

and moreover, f is continuous because of the property

$$|\mathrm{dist}(x, B) - \mathrm{dist}(y, B)| \leq |x - y|, \qquad x, y \in \mathbb{R}^n, \tag{101}$$

which holds for any set $B \subset \mathbb{R}^n$. The proof is completed. $\square$

As a simple application of mollifiers, let us prove the following important result known as the fundamental lemma of calculus of variations, which is attributed to Paul du Bois-Reymond.

Lemma 27 (du Bois-Reymond). *Let $u \in L^1_{\text{loc}}(\Omega)$ and let*

$$\int_\Omega u\varphi = 0 \qquad \text{for all} \quad \varphi \in \mathscr{D}(\Omega). \tag{102}$$

Then $u = 0$ almost everywhere in Ω.

Proof. Since mollification (90) is the integration against a function from $\mathscr{D}(\Omega)$ for small $\varepsilon > 0$, it follows that $u_\varepsilon(x) = 0$ eventually for each $x \in \Omega$. Let $K \subset \Omega$ be a compact set. Then u_ε converges to u in $L^1(K)$, meaning that $u = 0$ almost everywhere in K. As $K \subset \Omega$ was an arbitrary compact set, we conclude that $u = 0$ almost everywhere in Ω. $\square$

In order to study differentiability properties of u_ε, we need to be able to differentiate an integral with respect to a parameter. The following result is appropriate for our purposes.

Theorem 28 (Leibniz rule, version 2). *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, and let $I \subset \mathbb{R}$ be an open interval. Suppose that $f : \Omega \times I \to \mathbb{R}$ is a function satisfying*

- *$f(\cdot, t) \in L^1(\Omega)$ for each fixed $t \in I$,*
- *$f(y, \cdot) \in C^1(I)$ for almost every $y \in \Omega$,*
- *There is $g \in L^1(\Omega)$ such that $|f_t(y, t)| \leq g(y)$ for almost every $y \in \Omega$ and for each $t \in I$, where f_t is the derivative of f with respect to $t \in I$.*

Then the function $F : I \to \mathbb{R}$ defined by

$$F(t) = \int_\Omega f(y, t)\,\mathrm{d}y \qquad (t \in I), \tag{103}$$

satisfies $F \in C^1(I)$, and

$$F'(t) = \int_\Omega f_t(y, t)\,\mathrm{d}y, \qquad t \in I. \tag{104}$$

Proof. First, we claim that

$$G(t) = \int_\Omega f_t(y, t)\,\mathrm{d}y, \qquad t \in I, \tag{105}$$

is continuous in t. Since $f_t(y, t) - f_t(y, s) \to 0$ as $|t - s| \to 0$ for almost every y, it suffices to bound $|f_t(y, t) - f_t(y, s)|$ by an integrable function, uniformly in s and t. But this is exactly what we have assumed in the third bulleted item.

Now let $a \in I$ be an arbitrary but fixed point. Since G is continuous on I, from the fundamental theorem of calculus we have

$$G(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t G(s)\,\mathrm{d}s, \tag{106}$$

which leads to

$$\begin{aligned} \int_\Omega f_t(y, t)\,\mathrm{d}y &= \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \int_\Omega f_t(y, s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \int_a^t f_t(y, s)\,\mathrm{d}s\,\mathrm{d}y \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \big(f(y, t) - f(y, a)\big)\,\mathrm{d}y \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega f(y, t)\,\mathrm{d}y, \end{aligned} \tag{107}$$

where we have used Fubini's theorem in the second equality and the fundamental theorem of calculus for almost every $y \in \Omega$ in the third equality. $\square$

Exercise 10. Show that the preceding theorem is true when Ω is an arbitrary complete measure space. Moreover, in the context of the theorem, replace the conditions on f by

- $f(\cdot, t) \in L^1(\Omega)$ for almost every $t \in I$,
- $f(y, \cdot)$ is absolutely continuous on I, for almost every $y \in \Omega$,
- $f_t \in L^1(\Omega \times I)$,

and prove that F is absolutely continuous on I and $F' = G$ almost everywhere on I, where G is as in (105).

Corollary 29. *In the context of mollification, cf.* (89) *and* (90)*, let* $u \in L^1_{\mathrm{loc}}(\Omega)$ *and let* $K \subset \Omega$ *be a compact set. Then for all sufficiently small* $\varepsilon > 0$*, we have* $u_\varepsilon \in C^\infty(K)$ *and*

$$\partial^\alpha u_\varepsilon(x) = \int_{\mathbb{R}^n} \partial^\alpha \rho_\varepsilon(x-y)u(y)\,\mathrm{d}y \qquad (x \in K), \tag{108}$$

for any $\alpha \in \mathbb{N}_0^n$.

Proof. What we need to show is for $\phi \in \mathscr{D}(B_\varepsilon)$ with $\varepsilon > 0$ sufficiently small,

$$\frac{\partial}{\partial x_i}\int_{\mathbb{R}^n} \phi(x-y)u(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}\phi(x-y)u(y)\,\mathrm{d}y \qquad (x \in K). \tag{109}$$

With $t = x_i$, the conditions are easily verified. For instance, we have

$$\big|\frac{\partial}{\partial x_i}\phi(x-y)u(y)\big| \leq |u(y)|\sup_{B_\varepsilon}|\partial_i\phi|, \tag{110}$$

which confirms the condition after the third bullet point. $\square$

Now we can prove the main result of this section, the result known as *Weyl's lemma.*

Lemma 30 (Weyl 1940)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be an open set, and let* $u \in H^1(\Omega)$ *satisfy*

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0, \qquad \text{for all} \quad \varphi \in \mathscr{D}(\Omega). \tag{111}$$

Then up to a modification on a set of measure zero, u *is harmonic in the classical sense. In particular, we have* $u \in C^\omega(\Omega)$.

Proof. For $\varepsilon > 0$, let u_ε be the mollified version of u, cf. (89) and (90). Let $K \subset \Omega$ be a compact set, and let $\varepsilon > 0$ be sufficiently small. Then from Corollary 29, we know that $u_\varepsilon \in C^\infty(K)$, and

$$\Delta u_\varepsilon(x) = \int \Delta\rho_\varepsilon(x-y)u(y)\,\mathrm{d}y \qquad (x \in K). \tag{112}$$

Developing this further, we get

$$\begin{aligned}\Delta u_\varepsilon(x) &= \int \Delta\rho_\varepsilon(x-y)u(y)\,\mathrm{d}y = \int \Delta_y\rho_\varepsilon(x-y)u(y)\,\mathrm{d}y \\ &= -\int \nabla_y\rho_\varepsilon(x-y)\cdot\nabla u(y)\,\mathrm{d}y = 0,\end{aligned} \tag{113}$$

where we have used integration by parts for strong derivatives in the second equality, and the property (111) in the last equality. We also have used Δ_y and ∇_y to indicate that the implied derivatives are with respect to the y variable. Hence the functions u_ε are harmonic in K.

On the other hand, Theorem 25b) tells us that $u_\varepsilon \to u$ in $L^1(K)$ as $\varepsilon \to 0$. From the mean value property, it is easy to see that $\{u_\varepsilon\}$ forms a Cauchy sequence in the uniform norm on any compact set contained in the interior of K. This shows that u_ε converges locally uniformly to some harmonic function w in the interior of K. But u_ε also converges to u in $L^1(K)$, which means that $u = w$ almost everywhere in K. As $K \subset \Omega$ was an arbitrary compact set, we conclude that $u = w$ almost everywhere in Ω, with w a harmonic function in Ω. □

Exercise 11. Show that if $u \in L^1_{\mathrm{loc}}(\Omega)$ satisfies

$$\int_\Omega u\Delta\varphi = 0, \qquad \text{for all} \quad \varphi \in \mathscr{D}(\Omega), \tag{114}$$

then u is harmonic in the classical sense.

8. Weak derivatives

Recall that for functions $u, v \in L^2_{\mathrm{loc}}(\Omega)$, we say $v = \partial_i u$ strongly in L^2 if for each compact set $K \subset \Omega$, there exists a sequence $\{\phi_k\} \subset C^1(K)$ such that $\phi_k \to u$ and $\partial_i \phi_k \to v$ in $L^2(K)$. Strong derivatives are defined in terms of *approximation*. In order to show that a particular function is strongly differentiable by using the definition directly, one needs to construct a suitable approximating sequence, cf. Example 17. On the other hand, in the process of showing that string derivatives are unique, in Lemma 18 we proved that strong derivatives satisfy an integration by parts formula, namely

$$\int_\Omega \varphi \partial_i u = -\int_\Omega u \partial_i \varphi, \qquad \varphi \in C^1_c(\Omega). \tag{115}$$

We can turn this around and introduce a new concept of derivative, which is *a priori* more general than strong derivatives.

Definition 31. For $u, v \in L^1_{\mathrm{loc}}(\Omega)$, we say $v = \partial_i u$ in the *weak sense*, or that v is a *weak derivative* of u, if

$$\int_\Omega v\varphi = -\int_\Omega u \partial_i \varphi, \tag{116}$$

for all $\varphi \in \mathscr{D}(\Omega)$.

Weak derivatives are defined in terms of *duality*. It is immediate from the du Bois-Reymond lemma that the weak derivatives are unique.

Example 32. a) Let us try to find the weak derivative of $u(x) = |x|$, $x \in \mathbb{R}$. We have

$$\begin{aligned} \int_{\mathbb{R}} |x|\varphi'(x)\,\mathrm{d}x &= -\int_{-\infty}^0 x\varphi'(x)\,\mathrm{d}x + \int_0^\infty x\varphi'(x)\,\mathrm{d}x \\ &= \int_{-\infty}^0 \varphi(x)\,\mathrm{d}x - \int_0^\infty \varphi(x)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}} \varphi(x)\,\mathrm{sgn}(x)\,\mathrm{d}x, \end{aligned} \tag{117}$$

for $\varphi \in \mathscr{D}(\mathbb{R})$, implying that $|x|' = \mathrm{sgn}(x)$ in the weak sense. Note that as expected, the result is the same as that of Example 17.

b) Suppose that $v \in L^1_{\mathrm{loc}}(\mathbb{R})$ is the weak derivative of sgn. Then we would have

$$\int_{\mathbb{R}} v(x)\varphi(x)\,\mathrm{d}x = -\int_{\mathbb{R}} \varphi'(x)\,\mathrm{sgn}(x)\,\mathrm{d}x = -\int_0^\infty \varphi'(x)\,\mathrm{d}x + \int_{-\infty}^0 \varphi'(x)\,\mathrm{d}x = 2\varphi(0), \tag{118}$$

for $\varphi \in \mathscr{D}(\mathbb{R})$. In particular, it is true for $\varphi \in \mathscr{D}(\mathbb{R} \setminus \{0\})$, which by the du Bois-Reymond lemma implies that $v = 0$ almost everywhere in $\mathbb{R} \setminus \{0\}$. This of course means that $v = 0$ almost everywhere in $\mathbb{R}$, and for such functions, the integral in the left hand side of (118) is equal to 0. Hence (118) cannot be satisfied if $\varphi(0) \neq 0$, meaning that the sign function is *not* weakly differentiable.[2]

The following theorem shows that in the L^2-context, strong and weak derivatives coincide.

Theorem 33 (Friedrichs 1944)**.** *Let $u, v \in L^2_{\text{loc}}(\Omega)$. Then $v = \partial_i u$ in the strong L^2-sense if and only if $v = \partial_i u$ in the weak sense.*

Proof. The integration by parts formula (115) that we proved in Lemma 18 shows that if $v = \partial_i u$ in the strong L^2-sense, then $v = \partial_i u$ also in the weak sense.

Let $v = \partial_i u$ in the weak sense, and let $K \subset \Omega$ be a compact set. We will employ the technique of mollifiers, cf. (89) and (90). Let u_ε and v_ε be the mollified versions of u and v, respectively. We know that $u_\varepsilon \to u$ and $v_\varepsilon \to v$ in $L^2(K)$ as $\varepsilon \to 0$. What remains is to show that $\partial_i u_\varepsilon \to v$ in $L^2(K)$ as $\varepsilon \to 0$, but it follows from

$$\begin{aligned} \partial_i u_\varepsilon(x) &= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} \rho_\varepsilon(x-y) u(y) \, \mathrm{d}y = -\int_{\mathbb{R}^n} \frac{\partial}{\partial y_i} \rho_\varepsilon(x-y) u(y) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \rho_\varepsilon(x-y) v(y) \, \mathrm{d}y = v_\varepsilon(x), \end{aligned} \tag{119}$$

where in the third equality we have used the fact that v is the weak derivative of u. □

Definition 34. We define the *Sobolev space* $W^{1,2}(\Omega)$ as

$$W^{1,2}(\Omega) = \{u \in L^2(\Omega) : \partial_i u \in L^2(\Omega), \, i = 1, \ldots, n\}, \tag{120}$$

and equip it with the norm $\|\cdot\|_{H^1}$.

Theorem 35 (Meyers-Serrin 1964)**.** *For $\Omega \subset \mathbb{R}^n$ open, $C^\infty(\Omega) \cap H^1(\Omega)$ is dense in $W^{1,2}(\Omega)$. In particular, we have $H^1(\Omega) = W^{1,2}(\Omega)$.*

Proof. Let $u \in W^{1,2}(\Omega)$, and let $\varepsilon > 0$. We will show that there exists $\phi \in C^\infty(\Omega)$ such that $\|u - \phi\|_{H^1} \leq \varepsilon$. Consider a sequence $\{\Omega_k\}$ of bounded domains, such that $\Omega = \bigcup_k \Omega_k$ and $\overline{\Omega}_k \subset \Omega_{k+1}$ for $k = 1, 2, \ldots$. Moreover, for each k, let χ_k be a smooth nonnegative function satisfying $\operatorname{supp} \chi_k \subset \Omega_{k+2} \setminus \Omega_k$, and globally, $\sum_k \chi_k \equiv 1$ in Ω. Then for each k, we define $\phi_k = (\chi_k u)_{\varepsilon_k}$ by mollification, with $\varepsilon_k > 0$ so small that $\operatorname{supp} \phi_k \subset \Omega_{k+3} \setminus \Omega_{k-1}$ (with the convention $\Omega_0 = \varnothing$) and $\|\phi_k - \chi_k u\|_{H^1} \leq \varepsilon / 2^k$. This is possible because $\chi_k u \in W^{1,2}(\Omega)$ and $\partial_i \phi_k = (\partial_i(\chi_k u))_{\varepsilon_k}$. Finally, we define $\phi = \sum_k \phi_k$. There is no issue of convergence because the sum is locally finite. We have

$$\|u - \phi\|_{H^1} \leq \| \textstyle\sum_k (\chi_k u - \phi_k) \|_{H^1} \leq \textstyle\sum_k \|\chi_k u - \phi_k\|_{H^1} \leq \varepsilon, \tag{121}$$

which establishes the proof. □

9. Boundary values of weak solutions

To summarize what we have accomplished so far on the Dirichlet problem with the Sobolev space approach, for any given $g \in H^1(\Omega)$ with $\Omega \subset \mathbb{R}^n$ a bounded domain, we have constructed a harmonic function $u \in H^1(\Omega)$ satisfying $u - g \in H^1\mathscr{X}$. Recall from (40) that $\mathscr{X}$ is a linear space satisfying

$$\mathscr{D}(\Omega) \subset \mathscr{X} \subset C^1_0(\Omega) \equiv \{u \in C^1(\Omega) \cap C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}, \tag{122}$$

and from Definition 15 that $H^1\mathscr{X}$ is the closure of $\mathscr{X} \cap H^1(\Omega)$ in $H^1(\Omega)$. We also know from Weyl's lemma that u is harmonic in the classical sense in Ω. From now on, we will only consider the case $\mathscr{X} = \mathscr{D}(\Omega)$, and hence $H^1\mathscr{X} = H^1_0(\Omega)$. This is because the choice

[2] However, we have $\operatorname{sgn}' = 2\delta$ in the sense of distributions. From this perspective, the reason why sgn is not weakly differentiable is that *by definition* weak derivatives are locally integrable functions and δ is not a locally integrable function.

$\mathscr{X} = \mathscr{D}(\Omega)$ is simpler and already enough to guarantee harmonicity of u in the interior of the domain. Moreover, although we do not give a proof, it is true that $H^1\mathscr{X}$ does not depend on $\mathscr{X}$, as long as it satisfies (122).

The condition $u - g \in H^1_0(\Omega)$ is supposed to be a generalized form of the Dirichlet boundary condition $(u-g)|_{\partial\Omega} = 0$. We want to clarify what it would mean, at least when $\partial\Omega$ is not so irregular. To get some insight, let us consider the one dimensional case first.

Lemma 36. *Let $u \in H^1_0(\Sigma)$, with $\Sigma = (0,1)$. Then there is $w \in C(\bar{\Sigma})$ with $w(0) = w(1) = 0$ such that $u = w$ almost everywhere in Σ.*

Proof. There exists a sequence $\{u_k\} \subset \mathscr{D}(\Sigma)$ such that $u_k \to u$ in H^1. From the fundamental theorem of calculus, for $v \in \mathscr{D}(\Sigma)$ and for $0 < h < 1$ we have

$$v(h) = \int_0^h v'(t)\,\mathrm{d}t, \tag{123}$$

which implies that

$$|v(h)|^2 \leq h \int_0^h |v'(t)|^2\,\mathrm{d}t \leq h\|v'\|^2_{L^2}. \tag{124}$$

Applying this inequality to the differences $u_j - u_k$, we conclude that $\{u_k\}$ is Cauchy in the uniform norm on I and that $u_k \to w$ uniformly for some $w \in C(I)$. This means that $u = w$ almost everywhere. We want to see if the boundary value $w(0)$ can be defined. By continuity, we have

$$|w(h)| \leq \sqrt{h}\|u'\|_{L^2} \leq \sqrt{h}\|u\|_{H^1}, \tag{125}$$

and so

$$w(0) = \lim_{h\to 0} w(h) = 0, \tag{126}$$

establishing the lemma. □

Now we look at the two dimensional case, where a new phenomenon arises.

Lemma 37. *Let $\Sigma = (0,1)$ and $Q = \Sigma \times \Sigma$. For $0 < h < 1$, define $\gamma_h : \mathscr{D}(Q) \to \mathscr{D}(\Sigma)$ by $(\gamma_h\varphi)(x) = \varphi(x,h)$. Then γ_h can be uniquely extended to a bounded map $\gamma_h : H^1_0(Q) \to L^2(\Sigma)$, and for $u \in H^1_0(Q)$, we have $\gamma_h u \to 0$ in $L^2(\Sigma)$ as $h \to 0$.*

Proof. For $v \in \mathscr{D}(Q)$ and for $0 < h < 1$ we have

$$v(x,h) = \int_0^h \partial_y v(x,t)\,\mathrm{d}t, \tag{127}$$

which implies that

$$|v(x,h)|^2 \leq h \int_0^h |\partial_y v(t)|^2\,\mathrm{d}t \leq h \int_0^1 |\partial_y v(x,t)|^2\,\mathrm{d}t, \tag{128}$$

and upon integrating over x, that

$$\int_0^1 |v(x,h)|^2\,\mathrm{d}x \leq h \int_Q |\partial_y v(x,t)|^2\,\mathrm{d}t\,\mathrm{d}x \leq h\|\nabla v\|^2_{L^2(Q)}. \tag{129}$$

This means that $\|\gamma_h v\|_{L^2(\Sigma)} \leq \sqrt{h}\|v\|_{H^1(Q)}$ and that γ_h can be uniquely extended to a bounded map $\gamma_h : H^1_0(Q) \to L^2(\Sigma)$. □

The map γ_h in the preceding lemma is called the *trace map*, in the sense that functions defined on Q leave their traces on the lower dimensional manifold $\Sigma \times \{h\}$. Then the *boundary trace* $\gamma_0 u$ of u is defined in terms of the limit $\gamma_h u$ as $h \to 0$.

Example 38. Let $u \in \mathscr{D}(Q)$ be a function with $\gamma_h u \not\equiv 0$ for some $0 < h < 1$, and let $\phi \in \mathscr{D}(\mathbb{R})$ be a function satisfying $\phi(0) = 1$ and $0 \leq \phi \leq 1$. Then $v_k(x, y) = u(x, y)[1 - \phi(k(y - h))]$ satisfies $v_k \in \mathscr{D}(Q)$ and $\gamma_h v_k = 0$. Moreover, it is easy to see that $v_k \to u$ in $L^2(Q)$, because the area of the region on which v_k differs from u shrinks to 0. This shows that the trace map γ_h cannot be extended to $L^2(Q)$ as a continuous map, because $\gamma_h v_k = 0$ for all k, while $\gamma_h u_k = \gamma_h u \not\equiv 0$ for the constant sequence $u_k = u$.

Example 39. Let $\phi \in \mathscr{D}(\mathbb{R})$ be an even function with $\phi(0) = 1$, and let $u(\rho, \varphi, z) = \phi(\rho/z)$ be defined in the region $\{0 < z < 1\} \subset \mathbb{R}^3$ in cylindrical coordinates. Then $\gamma_h u = u|_{z=h}$ satisfies $\|\gamma_h u\|_{L^2(\mathbb{R}^2)} \to 0$ as $h \to 0$, because

$$\|\gamma_h u\|^2_{L^2(\mathbb{R}^2)} = 2\pi \int_0^\infty |\phi(\rho/h)|^2 \rho \,\mathrm{d}\rho = 2\pi h^2 \int_0^\infty |\phi(t)|^2 \,t \,\mathrm{d}t. \tag{130}$$

However, $(\gamma_h u)(0) = 1$ for all $h > 0$, hence $\gamma_h u$ does not go to 0 pointwise. This is an example where the boundary trace vanishes in the L^2-sense, but does not vanish pointwise. Moreover, we have $u \in H^1(\{0 < z < 1\})$, since

$$\int_0^\infty |\partial_\rho u|^2 \rho \,\mathrm{d}\rho = z^{-2} \int_0^\infty |\phi'(\rho/z)|^2 \rho \,\mathrm{d}\rho = \int_0^\infty |\phi'(t)|^2 \,t \,\mathrm{d}t, \tag{131}$$

and

$$\int_0^\infty |\partial_z u|^2 \rho \,\mathrm{d}\rho = z^{-4} \int_0^\infty |\phi'(\rho/z)|^2 \rho^3 \,\mathrm{d}\rho = \int_0^\infty |\phi'(t)|^2 \,t^3 \,\mathrm{d}t. \tag{132}$$

Exercise 12. Find a function $u \in H^1_0(\mathbb{H})$ where $\mathbb{H} \subset \mathbb{R}^2$ is the upper half plane, whose boundary trace vanishes in the L^2-sense, but does not vanish pointwise.

The general case is not more complicated than the two dimensional case.

Theorem 40. *Let $Q = (0, 1)^n$ and $\Sigma_h = (0, 1)^{n-1} \times \{h\}$. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\Phi : \bar{Q} \to \bar{\Omega}$ be an injective C^1 map, satisfying $\Phi(Q) \subset \Omega$ and $\Phi(\Sigma_0) \subset \partial\Omega$. With $\Gamma_h = \Phi(\Sigma_h)$ for $0 < h < 1$, define the trace map $\gamma_h : \mathscr{D}(\Omega) \to C(\Gamma_h)$ by $\gamma_h \varphi = \varphi|_{\Gamma_h}$. Then γ_h can be uniquely extended to a bounded map $\gamma_h : H^1_0(\Omega) \to L^2(\Gamma_h)$, and moreover, for $u \in H^1_0(\Omega)$ we have $\|\gamma_h u\|_{L^2(\Gamma_h)} \to 0$ as $h \to 0$.*

Proof. Let $X = \{v \in C^1(\bar{Q}) : v|_{\Sigma_0} = 0\}$, and define $\hat{\gamma}_h : X \to C(\Sigma_h)$ by $\hat{\gamma}_h v = v|_{\Sigma_h}$. Then as in the preceding lemma, we have $\|\hat{\gamma}_h v\|_{L^2(\Sigma_h)} \leq \sqrt{h} \|\nabla v\|_{L^2(Q)}$ for $v \in X$. Now let $u \in \mathscr{D}(\Omega)$. Then the pull-back $\hat{u} = \Phi^* u$ defined by $\hat{u}(\hat{x}) = u(\Phi(\hat{x}))$ satisfies $\hat{u} \in X$. Moreover, from the transformation properties of the first derivatives, we have

$$\|\nabla \hat{u}\|_{L^2(Q)} \leq c \|\nabla u\|_{L^2(\Omega)}, \qquad \text{where} \quad c = \sup_Q |\det D\Phi|^{-\frac{1}{2}} |D\Phi|, \tag{133}$$

and $|D\Phi|$ is the spectral norm of the Jacobian matrix $D\Phi$. We also have

$$\|\gamma_h u\|_{L^2(\Gamma_h)} \leq c' \|\hat{\gamma}_h \hat{u}\|_{L^2(\Sigma_h)}, \tag{134}$$

where c' depends only on the Jacobian $D\Phi$. Combining all three estimates, we infer

$$\|\gamma_h u\|_{L^2(\Gamma_h)} \leq C\sqrt{h} \|\nabla u\|_{L^2(\Omega)}, \tag{135}$$

and the theorem follows. □

Finally, we include a complementary result which basically says that if a function $u \in H^1_0(\Omega)$ is continuous at a boundary point $z \in \partial\Omega$, then $u(z) = 0$.

Lemma 41 (Nirenberg 1955)**.** *In the setting of the preceding theorem, let $u \in H^1_0(\Omega)$, and let u be continuous at $z \in \Phi(\Sigma_0)$. Then $u(z) = 0$.*

Proof. Without loss of generality we assume that $\Omega = \{x \in \mathbb{R}^n : |x| < 1,\, x_n > 0\}$ and $z = 0$. Let $u \in \mathscr{D}(\Omega)$, and with $h > 0$, let $E = B \times (0, h) \subset \Omega$ be a cylinder, where $B \subset \mathbb{R}^{n-1}$ is a ball centred at 0 whose volume is $|B| = h$. For $x \in E$, we have

$$|u(x)| = \Big| \int_0^{x_n} \partial_n u(x', t)\, \mathrm{d}t \Big| \leq \int_0^h |\partial_n u(x', t)|\, \mathrm{d}t, \tag{136}$$

where $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Now using the Cauchy-Bunyakowsky-Schwarz inequality and squaring, we get

$$|u(x)|^2 \leq h \int_0^h |\partial_n u(x', t)|^2\, \mathrm{d}t, \tag{137}$$

which, upon integrating along $x' = \text{const}$, gives

$$\int_0^h |u(x', t)|^2\, \mathrm{d}t \leq h^2 \int_0^h |\partial_n u(x', t)|^2\, \mathrm{d}t. \tag{138}$$

Then we integrate over $x' \in B$, and obtain

$$\int_E |u|^2 \leq h^2 \int_E |\partial_n u|^2 \leq h^2 \int_E |\nabla u|^2 = |E| \int_E |\nabla u|^2, \tag{139}$$

which means

$$\frac{1}{|E|} \int_E |u|^2 \leq \int_E |\nabla u|^2. \tag{140}$$

The same inequality is true for $u \in H^1_0(\Omega)$ by density, and the right hand side goes to 0 as $h \to 0$ by the fact that $u \in H^1(\Omega)$. Since u is continuous at 0, the left hand side goes to $|u(0)|^2$ as $h \to 0$, which proves the lemma. $\square$