MATH 580 ASSIGNMENT 3

DUE MONDAY NOVEMBER 11

1. Let $Q = (0,1)^n$ and let $Q_h = (h, 1-h)^n$. For $h > 0$ small, define the trace map $\gamma_h : C^1(Q) \to C(\partial Q_h)$ by $\gamma_h v = v|_{\partial Q_h}$.
   a) Prove that γ_h can be uniquely extended to a bounded map $\gamma_h : H^1(Q) \to L^2(\partial Q_h)$.
   b) Make sense of the boundary trace $\gamma_0 u = \lim\limits_{h\to 0} \gamma_h u$ in $L^2(\partial Q)$ for $u \in H^1(Q)$.
   c) Show that $\gamma_0 u = 0$ for $u \in H^1_0(Q)$.
   d) Let $u \in H^1_0(Q)$ and let u be continuous at 0. Show that $u(0) = 0$.
2. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $W^{1,1}_{\text{loc}}(\Omega)$ be the set of locally integrable functions whose (weak) derivatives are locally integrable (that is, in $L^1_{\text{loc}}(\Omega)$).
   a) Show that if $u, v \in W^{1,1}_{\text{loc}}(\Omega)$ and $uv, u\partial_i v + v\partial_i u \in L^1_{\text{loc}}(\Omega)$, then $uv \in W^{1,1}_{\text{loc}}(\Omega)$ and $\partial_i(uv) = u\partial_i v + v\partial_i u$.
   b) Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism between Ω and Ω'. Show that if $u \in W^{1,1}_{\text{loc}}(\Omega')$ then $v = u \circ \phi \in W^{1,1}_{\text{loc}}(\Omega)$ and $\partial_i v(x) = \sum_j \partial_i \phi_j(x)(\partial_j u)(\phi(x))$, where ϕ_j is the j-th component of ϕ, and $(\partial_j u)(\phi(x))$ is the evaluation of $\partial_j u$ at the point $\phi(x)$.
   c) Let $f \in C^1(\mathbb{R})$ with both f and f' bounded, and let $u \in W^{1,1}_{\text{loc}}(\Omega)$. Prove that $f \circ u \in W^{1,1}_{\text{loc}}(\Omega)$ and that $\partial_i(f \circ u) = (f' \circ u)\partial_i u$.
   d) Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ and let $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$ pointwise. Prove that $\partial_i u^+ = \theta(u)\partial_i u$ and $\partial_i u^- = \theta(-u)\partial_i u$ *a.e.*, where θ is the Heaviside step function. In particular, show that $|u| \in W^{1,p}(\Omega)$ if $u \in W^{1,p}(\Omega)$.
3. Let H be a (real) Hilbert space, and let H' be its dual, defined as the space of continuous linear functionals on H. Let us denote the inner product of H by $\langle \cdot, \cdot \rangle$. Observe that any $y \in H$ defines an element $f \in H'$ by $f(x) = \langle y, x \rangle$ for $x \in H$. This defines a map $J : H \to H'$. The *Riesz representation theorem* (for Hilbert spaces)[1] states that J is invertible, that is, any continuous linear functional on H can be realized through the inner product with an element of H. We would like to prove this theorem by using a variational method. Let $f \in H'$, and let
$$E(x) = \langle x, x \rangle - 2f(x), \qquad x \in H,$$
   and consider the problem of finding a minimizer of E over H.
   a) Show that a minimizing sequence for E exists and is Cauchy in H.
   b) Demonstrate that the limit minimizes E over H.

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[1]There is another result called Riesz representation theorem that is about representing linear functionals on a space of continuous functions as measures.

c) Denoting by $y \in H$ the minimizer, show that $\langle y, x\rangle = f(x)$ for all $x \in H$.
d) Finally, show that y depends continuously on f.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and consider the bilinear form
$$a(u,v) = \int_\Omega (a_{ij}\partial_i u \partial_j v + cuv),$$
where the repeated indices are summer over, and the coefficients a_{ij} and c are smooth functions on $\overline{\Omega}$, with a_{ij} satisfying the uniform ellipticity condition
$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \xi \in \mathbb{R}^n, \quad x \in \overline{\Omega},$$
for some constant $\lambda > 0$.
   a) Show that the mapping $A : H^1_0(\Omega) \to [H^1_0(\Omega)]'$, defined by $\langle Au, v\rangle = a(u,v)$, is bounded, where $\langle\cdot,\cdot\rangle$ is the duality pairing between $[H^1_0(\Omega)]'$ and $H^1_0(\Omega)$.
   b) Show that if $c \geq 0$ then
$$\langle Au, u\rangle \geq \alpha\|u\|^2_{H^1}, \qquad u \in H^1_0(\Omega),$$
for some constant $\alpha > 0$. Show also that the inequality is still true (with possibly different $\alpha > 0$) if c is slightly negative.
   c) Supposing that $c \geq 0$, show that given $f \in L^2(\Omega)$, there exists a unique function $u \in H^1_0(\Omega)$ satisfying $a(u,v) = \int_\Omega fv$ for all $v \in H^1_0(\Omega)$.
   d) Suppose that $u \in H^1_0(\Omega)$ is sufficiently smooth and satisfies $a(u,v) = \int_\Omega fv$ for all $v \in H^1_0(\Omega)$. What differential equation does u satisfy in Ω? Is $u = 0$ on $\partial\Omega$? In the language of variational methods, this is an *essential* boundary condition because it is incorporated into the space $H^1_0(\Omega)$.
5. Let a be the bilinear form as in the preceding question.
   a) Show that the mapping $A : H^1(\Omega) \to [H^1(\Omega)]'$, defined by $\langle Au, v\rangle = a(u,v)$, is bounded, where $\langle\cdot,\cdot\rangle$ is the duality pairing between $[H^1(\Omega)]'$ and $H^1(\Omega)$.
   b) Show that if $c > 0$ in $\overline{\Omega}$, then
$$\langle Au, u\rangle \geq \alpha\|u\|^2_{H^1}, \qquad u \in H^1(\Omega),$$
for some constant $\alpha > 0$.
   c) Supposing that the condition in b) holds, show that given $f \in L^2(\Omega)$, there exists a unique function $u \in H^1(\Omega)$ satisfying $a(u,v) = \int_\Omega fv$ for all $v \in H^1(\Omega)$.
   d) Suppose that $u \in H^1(\Omega)$ is sufficiently smooth and satisfies $a(u,v) = \int_\Omega fv$ for all $v \in H^1(\Omega)$. What differential equation does u satisfy in Ω? What boundary condition does u satisfy? This is a *natural* boundary condition because it arises from the equation u has to satisfy in the weak sense.
6. In the context of the preceding question, assuming $c \equiv 0$, prove that there exists a function $u \in H^1(\Omega)$ satisfying $a(u,v) = \int_\Omega fv$ for all $v \in H^1(\Omega)$ if and only if $\int f = 0$. Moreover, show that such a function is unique up to addition of a constant.