MATH 580 ASSIGNMENT 2

DUE MONDAY OCTOBER 7

1. Let Ω be an open subset of $\mathbb{R}^n$.
   (a) Show that if $u \in C^2(\Omega)$ is harmonic in Ω then
   $$\int_{\partial B} \partial_\nu u = 0,$$
   for any ball B whose closure is contained in Ω. Here ∂_ν is the normal derivative.
   (b) Suppose that $u \in C^1(\Omega)$ and that for each $y \in \Omega$ there exists $r^* > 0$ such that
   $$\int_{\partial B_r} \partial_\nu u = 0,$$
   for all $0 < r < r^*$. Show that u is harmonic in Ω (Bôcher 1905).
2. We say $u \in C(\Omega)$ is *subharmonic* in Ω if for each $y \in \Omega$ there exists $r^* > 0$ such that
   $$u(y) \leq \frac{1}{|B_r|} \int_{B_r(y)} u, \qquad \forall r \in (0, r^*).$$
   Prove the following statements.
   (a) A function $u \in C^2(\Omega)$ is subharmonic in Ω iff $\Delta u \geq 0$ in Ω.
   (b) A function $u \in C(\Omega)$ is subharmonic in Ω iff for any closed ball $B \subset \Omega$ and any harmonic function v in a neighbourhood of B, $u \leq v$ on ∂B implies $u \leq v$ in B.
   (c) A function subharmonic in $\mathbb{R}^2$ and bounded from above must be constant. Is this statement true in $\mathbb{R}^n$ for $n \geq 3$?
3. Prove that the function u given by the Poisson formula for the Dirichlet problem on a ball, say, B_r, is harmonic in B_r for boundary data $g \in L^1(\partial B_r)$, and takes correct boundary values wherever g is continuous.
4. (a) Find the region of $\mathbb{R}^2$ in which the power series $\sum_n x_1^n x_2^{n!}$ is absolutely convergent.
   (b) The *domain of convergence* of a power series is the interior of the region in which the series converges absolutely. Exhibit a two-variable real power series whose domain of convergence is the unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$.
   (c) Show that if $\Omega \subset \mathbb{R}^2$ is a domain of convergence of some power series centred at 0, then Ω is reflection symmetric with respect to the coordinate axes, and $\{(\log x_1, \log x_2) : x \in \Omega,\, x_1 > 0,\, x_2 > 0\}$ is a convex domain.
5. (Poincaré 1887) In this exercise, we will implement Poincaré's *method of sweeping out* (*méthode de balayage*) to solve the Dirichlet problem. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $g \in C(\Omega)$. Suppose that $u_0 \in C(\overline{\Omega})$ is a function subharmonic in Ω and

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$u_0 = g$ on $\partial\Omega$. The idea is to iteratively improve the initial approximation u_0 towards a harmonic function by solving the Dirichlet problem on a suitable sequence of balls.

(a) Show that there exist countably many open balls B_k such that $\Omega = \bigcup_k B_k$.

(b) Consider the sequence $B_1, B_2, B_1, B_2, B_3, B_1, \ldots$, so that each B_k is occurring infinitely many times, and let us reuse the notation B_k to denote the k-th member of this sequence. Then we define the functions $u_1, u_2, \ldots \in C(\overline{\Omega})$ by the following recursive procedure: For $k = 1, 2, \ldots$, put $u_k = u_{k-1}$ in $\Omega \setminus B_k$, and let u_k be the solution of $\Delta u_k = 0$ in B_k with the boundary condition $u_{k-1}|_{\partial B_k}$. Prove that $u_k \to u$ locally uniformly in Ω, for some $u \in C^\infty(\Omega)$ that is harmonic in Ω.

(c) Show that if there exists $v \in C(\overline{\Omega})$ satisfying $\Delta v = 0$ in Ω and $v = g$ on $\partial\Omega$, then indeed $u = v$, where u is the function we constructed in (b). So if there exists a solution, then our method would produce the same solution. However, we want to demonstrate existence without any prior assumption on existence.

(d) Prove that if there exists a barrier at $z \in \partial\Omega$, then $u(x) \to g(z)$ as $\Omega \ni x \to z$, where u is the function we constructed in (b). Recall that a function $\varphi \in C(\overline{\Omega})$ is called a *barrier for* Ω *at* $z \in \partial\Omega$ if
- φ is subharmonic in Ω,
- $\varphi(z) = 0$,
- $\varphi < 0$ in $\overline{\Omega} \setminus \{z\}$.

We call the boundary point $z \in \partial\Omega$ *regular* if there is a barrier for Ω at $z \in \partial\Omega$.

(e) Assuming that all boundary points are regular, this procedure reduces the Dirichlet problem into the problem of constructing a subharmonic function u_0 with $u_0|_{\partial\Omega} = g$. Instead of constructing such u_0 for the given g directly, let us approximate g by functions for which such a construction is simpler. Show that if $\{v_j\} \subset C(\overline{\Omega})$ is a sequence with $\Delta v_j = 0$ in Ω and $v_j \to g$ uniformly on $\partial\Omega$, then there exists a function $u \in C(\overline{\Omega})$ satisfying $\Delta u = 0$ in Ω and $u = g$ on $\partial\Omega$.

(f) Show that any polynomial can be written as the difference of two subharmonic functions in Ω. Hence it suffices to extend g into a continuos function on $\overline{\Omega}$, and approximate the resulting function by polynomials (explain why). State what standard results we need in order to realize this.

6. Let Ω be a bounded domain in $\mathbb{R}^n$.

(a) Show that if the Dirichlet problem in Ω is solvable for any boundary condition $g \in C(\partial\Omega)$, then each boundary point $z \in \partial\Omega$ admits a barrier.

(b) Why is regularity of a boundary point a local property? In other words, if $z \in \partial\Omega$ is regular, and if Ω' is a domain that coincides with Ω in a neighbourhood of z (hence in particular $z \in \partial\Omega'$), then is z also regular as a point on $\partial\Omega'$?

7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u \in C^2(\Omega)$ satisfy

$$E_*(u) := \int_\Omega (|\nabla u|^2 + |u|^2) < \infty.$$

Prove the followings.

(a) If $\Delta u = u$ in Ω, then $E_*(u + v) > E_*(u)$ for all nontrivial $v \in \mathscr{D}(\Omega)$.

(b) Conversely, if $E_*(u + v) \geq E_*(u)$ for all $v \in \mathscr{D}(\Omega)$, then $\Delta u = u$ in Ω.