MATH 580 ASSIGNMENT 1

DUE MONDAY SEPTEMBER 23

1. Exercises 2, 4, 6, 7, 11 from the lecture notes *Harmonic functions*.
2. Prove that the space of harmonic functions on an open set $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is infinite dimensional.
3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the boundary value problem
$$\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 1 \quad \text{on } \partial\Omega. \tag{$*$}$$
Prove the followings.
   a) Any solution of $(*)$ in $C^2(\Omega) \cap C(\bar{\Omega})$ with $f(u) = u^m$ where $m \in \mathbb{N}$ is odd, must satisfy $0 \leq u \leq 1$ in $\bar{\Omega}$, and is unique.
   b) The only solution of $(*)$ in $C^2(\Omega) \cap C(\bar{\Omega})$ with $f(u) = u - u^{-1}$ is $u \equiv 1$.
4. Let $\phi : (0, 1] \to \mathbb{R}$ be a continuous nonnegative function satisfying
$$\int_\varepsilon^1 \phi(r) r^{n-1} \mathrm{d}r \leq M,$$
for any $\varepsilon > 0$, with a fixed constant M. Define $f(x) = \phi(|x|)$ for $x \in B_1$ where $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$. Prove in complete detail that $f \in L^1(B_1)$.
5. Let $u \in \mathscr{C}^2(\mathbb{R}^n)$ be an entire harmonic function in $\mathbb{R}^n$. Prove the followings.
   (a) If $u \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$ then $u \equiv 0$.
   (b) If u satisfies $u(x) \geq -C(1 + |x|)^m$ for some constants C and $m \in \mathbb{N}$, then u is a polynomial of degree less or equal to m.
   (c) Any tangent hyperplane to the graph of u intersects the graph more than once.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.

Date: Fall 2013.