Minimal Surfaces

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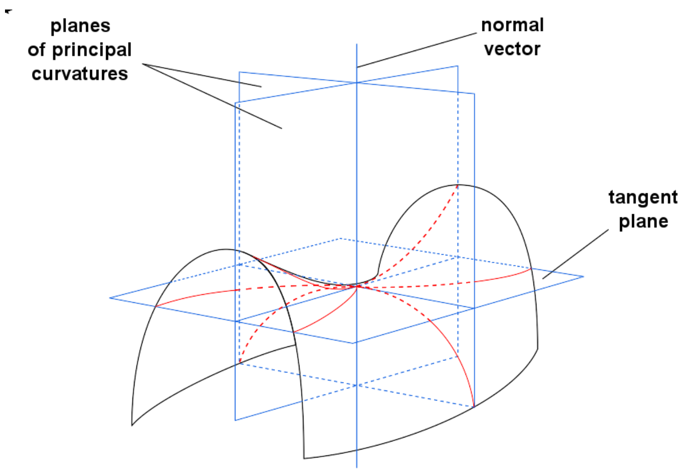
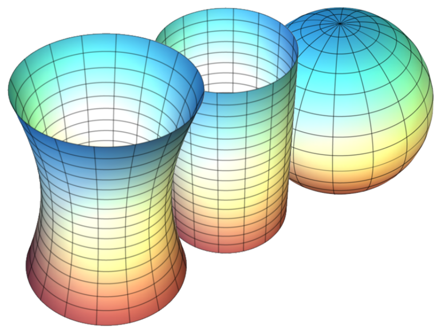
Intuitively, a Minimal Surface is a surface that has minimal area, locally. First, we will give a mathematical definition of the minimal surface. Then, we shall give some examples of Minimal Surfaces to gain a mathematical understanding of what they are and finally move on to a generalization of minimal surfaces, called Willmore Surfaces. The reason for this is that Willmore Surfaces are an active and important field of study in Differential Geometry. We will end with a presentation of the Willmore Conjecture, which has recently been proved and with some recent work done in this area. Until we get to Willmore Surfaces, we assume that we are in $\mathbb{R}^3$.

Definition 1: The two **Principal Curvatures**, k_1 & k_2 at a point $p \in S$, $S \subset \mathbb{R}^3$ are the eigenvalues of the shape operator at that point. In classical Differential Geometry, k_1 & k_2 are the maximum and minimum of the **Second Fundamental Form**. The principal curvatures measure how the surface bends by different amounts in different directions at that point. Below is a saddle surface together with normal planes in the directions of principal curvatures.



Definition 2: The **Mean Curvature** of a surface S is an extrinsic measure of curvature; it is the average of it's two principal curvatures: $\mathbb{H} \equiv \frac{1}{2}(k_1 + k_2)$.

Definition 3: The **Gaussian Curvature** of a *point* on a surface S is an intrinsic measure of curvature; it is the product of the principal curvatures: $\mathbb{K} \equiv k_1 k_2$ of the given point. The Gaussian Curvature is intrinsic in the sense that it's value depends only on how distances are measured on the surface, not on the way it is isometrically embedded in space. Changing the embedding will not change the Gaussian Curvature. Below, we have a hyperboloid (negative Gaussian Curvature), a cylinder (zero Gaussian curvature) and a sphere (positive Gaussian curvature).



Definition 4: A surface $S\subset \mathbb{R}^3$ is **Minimal** if and only if it's mean curvature is 0. An equivalent statement is that a surface $S\subset \mathbb{R}^3$ is **Minimal** if and only if every point $p \in S$ has a neighbourhood with least-area relative to its boundary. Yet another equivalent statement is that the surface is **Minimal** if and only if it's principal curvatures are equal in magnitude but necessarily differ by sign.

Definition 5: For at a point $p \in S\subset \mathbb{R}^3$, if $k_1 = k_2$, then p is called an umbilical point of S. In particular, in a plane, all point are umbilical points. In addition, it can be proved that if all points of a connected surface are umbilical, then that surface is entirely contained in a sphere or in a plane.

We shall now give some examples of minimal surfaces. A trivial minimal surface is the plane itself. Intuitively, it is very easy to see why it has minimal area locally (and even globally). The first non-trivial minimal surface is the **Catenoid**, it was discovered and proved to be minimal by Leonhard Euler in 1744. The Catenoid has parametric equations:

$x = c\cosh\frac{v}{c}\cos u$

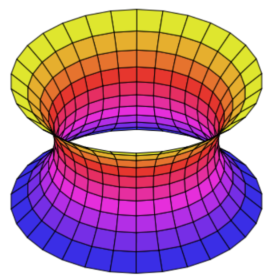
$y = c\cosh\frac{v}{c}\sin u$

$z = v$

It's principal curvatures are:

$k_1 = \frac{1}{c}(\cosh\frac{v}{c})^{-1}$

$k_2 = -\frac{1}{c}(\cosh\frac{v}{c})^{-1}$



In 1776, Jean Baptiste Meusnier discovered the **Helicoid** and proved that it was also a minimal surface. The name is derived from the helix; for every point on the helicoid, there exists a helix in the helicoid which passes through that point. The Helicoid shares some interesting properties with the Catenoid, such as the ability to “bend” one into the other without “tearing” the surface. The Helicoid has parametric equations:

$x = \rho \cos \alpha\theta$

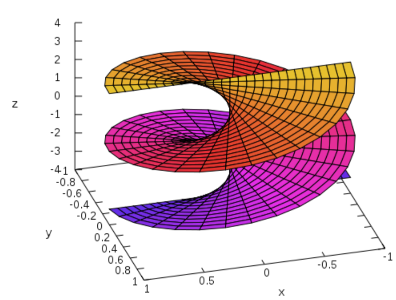
$y = \rho \sin \alpha\theta$

$z = \theta$

where ρ & $\theta \in (-\infty, \infty)$, while α is a constant. It’s principal curvatures are:

$k_1 = \frac{1}{(1+p^2)}$

$k_2 = -\frac{1}{(1+p^2)}$



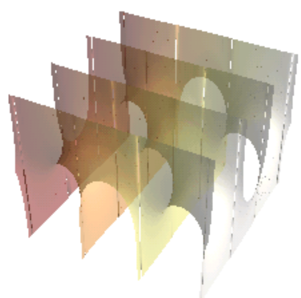
In 1834, Heinrich Ferdinand Scherk discovered two other minimal surfaces, which are now named **Scherk's First Surface** and **Scherk's Second Surface**. The first surface is doubly periodic while the second is only singly periodic. The surfaces are conjugate to one another. Below are both surfaces, together with their parametrizations.

First surface:

$x(u,v) = a\,u$

$y(u,v) = a\,v$

$z(u,v) = a\,\ln(\cos u \sec v)$



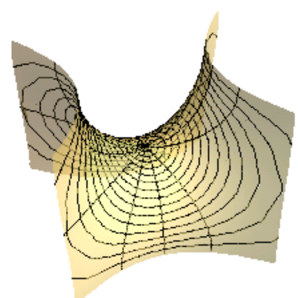
(plotted for u from -5 to 5 and v from -5 to 5)

Second surface:

$x(r,v) = a\,\log(\frac{r^2+2r\cos v+1}{r^2-2r\cos v+1})$

$y(r,v) = a\,\log(\frac{r^2-2r\sin v+1}{r^2+2r\sin v+1})$

$z(r,v) = 2a\,\tan^{-1}(\frac{2r^2\sin 2v}{r^4-1})$



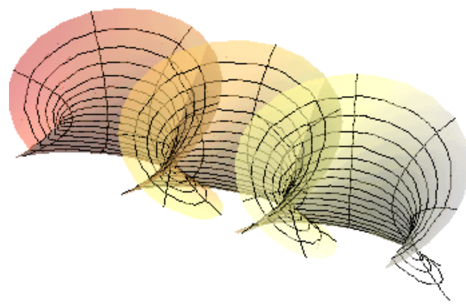
(plotted for r from 0 to 1 and v from 0 to 2π)

In 1855, as part of his work on minimal regular surfaces, the Belgian mathematician Eugene Charles Catalan created a minimal surface containing an entire family of parabolae, now called the **Catalan Minimal Surface**.

$x(u,v) = a(u - \sin(u)\cosh(v))$

$y(u,v) = a(1 - \cos(u)\cosh(v))$

$z(u,v) = 4a\sin\left(\frac{u}{2}\right)\sinh\left(\frac{v}{2}\right)$



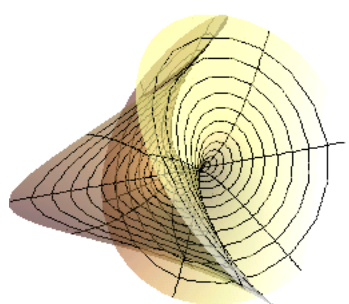
(plotted for u from 0 to 6π and v from -2 to 2)

In 1864, Alfred Enneper discovered a minimal surface conjugate to itself, now called the **Enneper Surface**. Below is the surface, together with it's parametric equations.

$x(r,\phi) = r\cos\phi - \frac{1}{3}r^3\cos 3\phi$

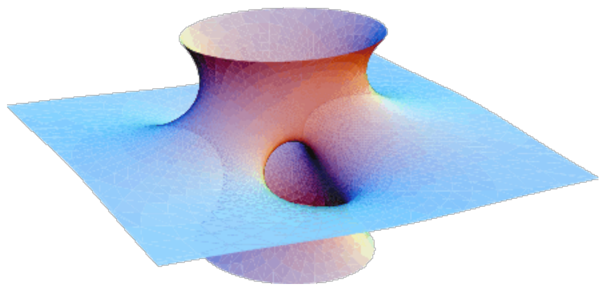
$y(r,\phi) = -\frac{1}{3}r(r^2\sin 3\phi + 3\sin\phi$

$z(r,\phi) = r^2\cos 2\phi$



(plotted for r from 0 to 2 and ϕ from 0 to 2π)

Until now, we have only presented minimal surfaces whose parametrizations are simple. A more challenging one is **Costa's Minimal Surface**, discovered by Celso Costa in 1982. Topologically, it is a thrice-punctured ring torus. Costa's surface also belongs to D_4, the dihedral group of symmetries, in Algebra. While Costa is the first to have imagined the surface, parametrizing it would turn out to be very difficult and the first parametrization had to wait until 1996. We do not present it here as it is very complicated and beyond the scope of this paper.



On the other hand, while intuitively, the sphere minimizes area locally, it does not have zero mean curvature and is therefore not a minimal surface. The ring torus is another example of a surface that is not minimal. We will not give the principal curvatures or mean curvatures for these two, since the calculations are more complicated. However, while the ring torus is not a minimal surface, there are two other kinds of tori which are minimal surfaces, the Clifford Torus and the Otsuki Torus. The Otsuki Torus' parametrization is too complicated so we omit this surface here. However, many properties of Otsuki tori are to be found in [2]. We present the Clifford Torus now and use it as an introduction to Willmore surfaces. We now leave $\mathbb{R}^3$.

The **Clifford Torus** is a special kind of Torus in $\mathbb{R}^4$. Since topologically, $\mathbb{C}^2 \cong \mathbb{R}^4$, we can view the Clifford Torus as sitting inside $\mathbb{C}^2$. Every point of the Clifford Torus lies at a fixed distance from the origin and it can therefore also be embedded in $\mathbb{S}^3$. In $\mathbb{R}^4$, the Clifford Torus is:

$$\mathbb{S}^1 \times \mathbb{S}^1 = \{(\cos\theta \sin\theta \cos\phi \sin\phi) \,|\, 0 \le \theta < 2\pi,\, 0 \le \phi < 2\pi\}$$

The Clifford Torus is important as it is used to define the Willmore Conjecture, an important conjecture in the Calculus of Variations, which uses it's stereographic projection into $\mathbb{R}^3$. We now present this conjecture.

A **Willmore Surface** is a generalization of a minimal surface. Willmore surfaces Σ in $\mathbb{R}^3$ are stationary immersed submanifolds for the **Willmore Functional** or **Willmore Energy**. The Willmore Energy is a way of measuring how much a surface in $\mathbb{R}^3$ deviates from a sphere. Essentially, it is a measure of area, taking it's minimum value at the round sphere. In biology, it is used to study cell membranes.

$$W(\Sigma) = \int_\Sigma H^2 dA$$

where H is the mean curvature of Σ and dA is the area form. The Willmore Functionals are the solutions to the Euler-Lagrange Equation:

$$\triangle H + 2H(H^2 - K) = 0$$

where K is the Gaussian Curvature of Σ. In the Calculus of Variations, the solutions of the Euler-Lagrange Equation are the functions for which a given functional is stationary. The minimal surfaces are special cases of Willmore functionals, and they are, therefore, also solutions to the Euler-Lagrange Equation. In the early 1960s, Willmore formulated the problem of minimizing the Willmore functional. However, smooth surfaces which are critical points of the Willmore functional with respect to normal variations have been studied since the first quarter of the previous century. First, it is easy to see that a natural lower bound for the Willmore functional is:

$$W \geq 4\pi$$

for immersed surfaces of any genus, with equality precisely for round embedded spheres. To get this result, we define $\Sigma_+ = \{x \in \Sigma \,|\, K \geq 0\}$, where $K : \Sigma \to R$ is the Gaussian curvature and we use the fact that $H^2 - K = \frac{1}{4}(k_1 - k_2)^2 \geq 0$. This gives:

$$W(\Sigma) \geq \int_{\Sigma_+} H^2 dA \geq \int_{\Sigma_+} K dA \geq 4\pi$$

In the case of equality, we clearly have $H^2 - K = 0$. We know that the only totally umbilic compact surfaces in $\mathbb{R}^3$ are round spheres, for which we clearly have equality (see definition 5). However, for surfaces of genus greater than zero, we expect a bound somewhat stronger than the one above. In particular, calculations of W for tori with various symmetries led Willmore to the following conjecture, in 1965:

The Willmore Conjecture: Given any smooth immersed torus in $\mathbb{R}^3$, the Willmore functional should be bound by the inequality

$$W \geq 2\pi^2$$

Equality is achieved above for the stereographic projection of the Clifford Torus into $\mathbb{R}^3$, given by:

$$(\theta, \phi) \longrightarrow ((\sqrt{2} + \cos\phi)\cos\theta, (\sqrt{2} + \cos\phi)\sin\theta, \sin\phi)$$

The conjecture has only very recently been proved, despite serious attempts to solve it. Willmore guessed the correct bound, but we had to wait 47 years for the actual proof. Li & Yau proved [6] a stronger lower bound than 4π and their work was later generalized by Peter Topping [5], in a complex Lemma involving also the area of the surface Σ. We do not state the entire Lemma here, or it's proof, but only the result of the stronger upper bound:

$$W \geq 4k\pi$$

where k is a constant, which varies depending on the surface Σ chosen. Particularly, the Willmore functional of a surface admitting self-intersections has an even stronger lower bound:

$$W \geq 8\pi$$

i.e. $k = 2$ for such surfaces. Now, in the Willmore Conjecture $W = 2\pi^2 < 8\pi$. Therefore, the Willmore Conjecture only needs to be proven for tori that are embedded. In addition, Rosenberg & Langevin [7] have proven, using the Gauss-Bonnet Formula:

$$W(\Sigma) \geq \int_{\Sigma_+} H^2 dA \geq \int_{\Sigma_+} K dA = \frac{1}{2}\int_{\Sigma} |K| dA \geq 8\pi$$

Therefore, the conjecture only needs to be proven for embedded tori which are isotopic to the stereographic projection of the Clifford Torus into $\mathbb{R}^3$, given above. The existence of a torus which attains this minimum value has been proven, but unfortunately, this was not sufficient in bringing down the conjecture. We end with a list of special cases in which the Willmore Conjecture has been proven, including the full result, which was proven in the beginning of this year.

1. In the early 1970s, Willmore [8] and independently Shiohama and Takagi proved the conjecture for tube tori with constant radius. A tube torus is formed by carrying a small circle around a closed space curve so that the centre moves along the curve and the plane of the circle is the normal plane to the curve at each point [9].

2. Hertrich-Jeromin and Pinkall generalized the result above (1) by allowing the radius of the circle to vary along the curve. This was done in a German Mathematics Journal which I cannot access (or read), but their work was mentionned by Willmore.

3. Langer and Singer proved the conjecture for tori of revolution.

4. The work of Li & Yau in finding a stronger lower bound was mentionned above and in [6]. In the same paper, they also introduced the concept of conformal volume and proved the conjecture for tori whose conformal structure are defined by lattices generated by certain classes of vectors. A special case of two vectors produces the Clifford torus.

5. Ros proved in [11] that any immersed torus in $\mathbb{S}^3$ which remains invariant when composed with the antipodal map of $\mathbb{S}^3$, produces, after stereographic projection, a Willmore surface with $W = 2\pi^2$. Therefore, Ros proved that there exists a Willmore surface reaching this bound, but not that this is the lowest any surface admitted by the Willmore functional can go. The result by Ros, we state formally, followed by a very brief proof, since the proof relies on other complicated results by Ros:

   (a) **Theorem [Ros, 1997]:** For any compact surface Σ in $\mathbb{S}^3$ of odd genus, which is antipodal invariant, we have $\int_\Sigma 1 + H^2 dA \geq 2\pi^2$. Equality holds if and only if Σ is the minimal Clifford Torus.

   (b) **Proof:** The genus of Σ is odd. In the same paper, Ros proved that the antipodal map preserves components in $\mathbb{S}^3 - \Sigma$. Therefore, the quotient surface $\Sigma' = \Sigma/\pm$ separates the projective space $\mathbb{P}^3$. From additional results also by Ros, $\int_\Sigma 1 + H^2 dA = 2\int_{\Sigma'} 1 + H^2 dA \geq 2\pi^2$, with equality holding only for the Clifford Torus.

6. In February 2012, Fernando Codá Marques and André Neves announced a full proof of the Willmore conjecture and published it pre-print on arXiv. The full proof is 96 pages long and relies on advanced notions such as homotopy.

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