# MATH 580 ASSIGNMENT 6

DUE WEDNESDAY DECEMBER 5

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and let

$$Lu = -a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu,$$

where $a_{ij}$ satisfies the uniform ellipticity condition, and all coefficients are smooth in $\overline{\Omega}$.

- The *Calderon-Zygmund estimate* (or the *elliptic estimate in* $L^p$)

$$\|u\|_{W^{k+2,p}(\Omega)} \leq C(\|Lu\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)}),$$

  holds for all $u \in C_c^\infty(\Omega)$, where the constant $C$ depends only on $\Omega$, $1 < p < \infty$ and $k \geq 0$.
- The *Schauder estimate*

$$\|u\|_{C^{k+2,\alpha}(\Omega)} \leq C(\|Lu\|_{C^{k,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)}),$$

  holds for all $u \in C_c^\infty(\Omega)$, where the constant $C$ depends only on $\Omega$, $k \geq 0$ and $0 < \alpha < 1$.

Assume that for any $f \in L^2(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega)$ satisfying $Lu = f$. As we have seen in the previous assignment, this is in particular guaranteed if the coefficients $b_i$ and $c$ are so that $L$ is strictly coercive in $H_0^1(\Omega)$. Moreover, we have the $L^2$-regularity result saying that if $f \in H^k(\Omega)$ then $u \in H^{k+2}(\Omega)$.

a) Using the elliptic $L^p$-estimate, prove the $L^p$-regularity result that if $f \in W^{k,p}(\Omega)$ then $u \in W^{k+2,p}(\Omega)$, where $1 < p < \infty$ and $k \geq 0$.

b) Using the Schauder estimate, prove the Hölder regularity result that if $f \in C^{k,\alpha}(\Omega)$ then $u \in C^{k+2,\alpha}(\Omega)$, where $k \geq 0$ and $0 < \alpha < 1$.

2. Consider the Dirichlet problem for the *biharmonic equation*

$$\Delta^2 u = f, \qquad u|_{\partial\Omega} = \partial_\nu u|_{\partial\Omega} = 0.$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^n$, where $f \in L^2(\Omega)$.

a) Define a natural concept of weak solutions in $H_0^2(\Omega)$, the latter space being the closure of $C_c^\infty(\Omega)$ in $H^2(\Omega)$. In particular, show that all classical solutions are weak solutions, and that sufficiently regular weak solutions are classical solutions.

b) Show that there exists a unique weak solution $u \in H_0^2(\Omega)$.

c) Prove that if $f \in H^k(\Omega)$ then $u \in H^{k+4}(\Omega)$.

d) What would the natural boundary conditions be for the biharmonic equation?

---

*Date*: Fall 2012.

3. Let $V \hookrightarrow H$ be Hilbert spaces, where the embedding is compact and dense, and let $A : V \to V^*$ be a self-adjoint and coercive operator. We are interested in the *abstract Schrödinger equation*
$$u' = iAu,$$
to be solved for $u : \mathbb{R} \to H$, with the initial condition $u(0) = g \in \mathrm{Dom}A$.
   a) Show that
   $$u(t) = e^{itA}g,$$
   satisfies $u \in C^1(\mathbb{R}, H)$, and is a solution.
   b) Show that for any solution $u$, the energy $E(t) = \|u(t)\|_H^2$ is preserved. In particular, the solution is unique.
   c) Show that $e^{itA}e^{isA} = e^{i(t+s)A}$ for $s, t \in \mathbb{R}$ and $\|e^{itA}v\|_H = \|v\|_H$ for $t \in \mathbb{R}$, i.e., the *Schrödinger propagators* $e^{itA}$ form a one-parameter unitary group.
   d) Solve the initial value problem
   $$u_t = iu_{xx}, \qquad u(x, 0) = g(x),$$
   on the interval $[0, \pi]$ with the homogeneous Dirichlet boundary conditions, where $g$ is a smooth function satisfying $g(0) = g(\pi) = 0$.
4. Consider the Cauchy problem for the wave equation $\partial_t^2 u - \Delta u = 0$ in $\mathbb{R}^n$. Suppose that $u : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}$ is its solution, and that the initial data vanish outside $\Omega \subset \mathbb{R}^n$. Show that $u(x, t) = 0$ if $|t| < \inf_{y \in \Omega} |x - y|$.
5. By way of examples, make a strong case against well-posedness of the Cauchy problem for the Laplace equation
$$\Delta u = 0 \quad \text{in } \{x \in \mathbb{R}^n : x_n > 0\}, \qquad u|_{x_n=0} = g, \quad \partial_n u|_{x_n=0} = f.$$