MATH 580 ASSIGNMENT 5

DUE THURSDAY NOVEMBER 22

1. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $W^{1,1}_{\text{loc}}(\Omega)$ be the set of locally integrable functions whose derivatives are locally integrable.
   a) (Product rule) Show that if $u, v \in W^{1,1}_{\text{loc}}(\Omega)$ and $uv, u\partial_i v + v\partial_i u \in L^1_{\text{loc}}(\Omega)$, then $uv \in W^{1,1}_{\text{loc}}(\Omega)$ and $\partial_i(uv) = u\partial_i v + v\partial_i u$.
   b) (Coordinate change) Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism between Ω and Ω'. Show that if $u \in W^{1,1}_{\text{loc}}(\Omega')$ then $v = u \circ \phi \in W^{1,1}_{\text{loc}}(\Omega)$ and $\partial_i v(x) = \sum_j \partial_i \phi_j(x)(\partial_j u)(\phi(x))$, where ϕ_j is the j-th component of ϕ, and $(\partial_j u)(\phi(x))$ is the evaluation of $\partial_j u$ at the point $\phi(x)$.
   c) (Chain rule) Let $f \in C^1_b(\mathbb{R}^1)$ and $u \in W^{1,1}_{\text{loc}}(\Omega)$. Prove that $f \circ u \in W^{1,1}_{\text{loc}}(\Omega)$ and that $\partial_i(f \circ u) = (f' \circ u)\partial_i u$.
   d) (Chain rule with piecewise smooth function) Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ and let $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. Show that $\partial_i u^+ = \theta(u)\partial_i u$ and $\partial_i u^- = \theta(-u)\partial_i u$ *a.e.*, where θ is the Heaviside step function. In particular, we have $u^+, u^-, |u| \in W^{1,1}_{\text{loc}}(\Omega)$.
2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary.
   a) Using Green's first identity, show that
   $$\|\partial_i u\|_{L^2(\Omega)} \leq \varepsilon \|u\|_{H^2(\Omega)} + C\varepsilon^{-1}\|u\|_{L^2(\Omega)},$$
   for any $\varepsilon > 0$ and $u \in H^2_0(\Omega)$, where $C > 0$ is a constant.
   b) Using an extension result, generalize the above inequality to $u \in H^2(\Omega)$, with $C > 0$ possibly depending on Ω, whose boundary is assumed to be sufficiently smooth.
3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary.
   a) Show that Ω satisfies the *cone condition*: There exists a right circular (open) cone $\mathscr{C}$ of height $h > 0$ and solid angle $\omega > 0$ at its vertex, such that for each point $x \in \overline{\Omega}$ there is a cone $\mathscr{C}_x \subset \Omega$ congruent to $\mathscr{C}$, with its vertex at x (in other words, one can place $\mathscr{C}$ in Ω by moving its vertex to x and by suitably rotating around x).
   b) For $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and $x \in \overline{\Omega}$, show that
   $$\omega |u(x)| \leq \int_{\mathscr{C}_x} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \,\mathrm{d}y + \frac{1}{h}\int_{\mathscr{C}_x} \frac{|u(y)|}{|x-y|^{n-1}} \,\mathrm{d}y.$$
   c) Using the result from the previous assignment on the Riesz potentials, prove the *Sobolev inequality*
   $$c\omega \|u\|_{L^q(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)} + \frac{1}{h}\|u\|_{L^p(\Omega)},$$

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for $u \in C^1(\Omega) \cap C(\overline{\Omega})$, where $1 \leq p < n$, $1 \leq q < \frac{np}{n-p}$, and $c > 0$ is a constant. Extend the inequality to $u \in W^{1,p}(\Omega)$ by density. Note that this approach misses the endpoint $q = \frac{np}{n-p}$, even though the Sobolev inequality is valid there. It is possible to remedy this at least for $p > 1$ by using improved estimates for the Riesz potentials.

4. Let H be a Hilbert space. Prove the followings.
   a) Let $f : H \to H$ be a Lipschitz continuous map satisfying
   $$\langle f(u) - f(v), u - v\rangle \geq \alpha \|u - v\|^2, \qquad u, v \in H,$$
   for some constant $\alpha > 0$. Then there is a unique $u \in H$ such that $f(u) = 0$. *Hint*: Consider the map $\phi(u) = u - \omega f(u)$, where $\omega > 0$ is a parameter to be adjusted.
   b) (Lax-Milgram lemma) Let $\ell \in H'$, and let $A : H \to H'$ be a bounded linear operator satisfying
   $$\langle Au, u\rangle \geq \alpha \|u\|^2, \qquad u \in H,$$
   for some constant $\alpha > 0$. Then there is a unique $u \in H$ such that $Au = \ell$.
5. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and consider the bilinear form
   $$a(u, v) = \int_\Omega (a_{ij}\partial_i u \partial_j v + b_i \partial_i uv + b'_i u \partial_i v + cuv),$$
   where the repeated indices are summer over, and the coefficients a_{ij}, b_i, b'_i, and c are smooth functions on $\overline{\Omega}$, with a_{ij} satisfying the uniform ellipticity condition
   $$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \xi \in \mathbb{R}^n, \quad x \in \overline{\Omega},$$
   for some constant $\lambda > 0$. To clarify the notation, we understand that a differential operator acts only on the function immediately following it, i.e., $\partial_i uv = (\partial_i u)v$.
   a) Show that the mapping $A : H^1_0(\Omega) \to [H^1_0(\Omega)]'$, defined by $\langle Au, v\rangle = a(u, v)$, is bounded, where $\langle \cdot, \cdot\rangle$ is the duality pairing between $[H^1_0(\Omega)]'$ and $H^1_0(\Omega)$.
   b) Show that if $c \geq 0$ then
   $$\langle Au, u\rangle \geq \alpha \|u\|^2_{H^1}, \qquad u \in H^1_0(\Omega),$$
   for some constant $\alpha > 0$. Show also that the inequality is still true (of course with possibly different $\alpha > 0$) if c is slightly negative.
   c) Supposing that $c \geq 0$, show that given $f \in L^2(\Omega)$, there exists a unique function $u \in H^1_0(\Omega)$ satisfying $a(u, v) = \int_\Omega fv$ for all $v \in H^1_0(\Omega)$.
   d) In the above setting, prove that $u \in H^{k+2}_{\text{loc}}(\Omega)$ if $f \in H^k(\Omega)$.
   e) Suppose that $u \in H^1_0(\Omega)$ is sufficiently smooth and satisfies $a(u, v) = \int_\Omega fv$ for all $v \in H^1_0(\Omega)$. What differential equation does u satisfy in Ω? Is $u = 0$ on $\partial\Omega$? This boundary condition is called *essential* because it is incorporated into the space $H^1_0(\Omega)$.
6. Let a be the bilinear form as in the preceding question.
   a) Show that the mapping $A : H^1(\Omega) \to [H^1(\Omega)]'$, defined by $\langle Au, v\rangle = a(u, v)$, is bounded, where $\langle \cdot, \cdot\rangle$ is the duality pairing between $[H^1(\Omega)]'$ and $H^1(\Omega)$.
   b) Show that if $\operatorname{div}(b + b') < c$ and $(b + b') \cdot \nu \geq 0$ where ν is the outer normal to the boundary $\partial\Omega$, then
   $$\langle Au, u\rangle \geq \alpha \|u\|^2_{H^1}, \qquad u \in H^1(\Omega),$$

for some constant $\alpha > 0$.

c) Supposing that the conditions in b) hold, show that given $f \in L^2(\Omega)$, there exists a unique function $u \in H^1(\Omega)$ satisfying $a(u,v) = \int_\Omega fv$ for all $v \in H^1(\Omega)$.

d) Assuming $b_i = b_i' = c \equiv 0$, prove that there exists a function $u \in H^1(\Omega)$ satisfying $a(u,v) = \int_\Omega fv$ for all $v \in H^1(\Omega)$ if and only if $\int f = 0$. Moreover, such a function is unique up to addition of a constant.

e) Suppose that $u \in H^1(\Omega)$ is sufficiently smooth and satisfies $a(u,v) = \int_\Omega fv$ for all $v \in H^1(\Omega)$. What differential equation does u satisfy in Ω? What boundary condition does u satisfy? This boundary condition is called *natural* because it arises from the equation u has to satisfy in the weak sense.

f) Suppose that we are given the differential equation

$$-a_{ij}\partial_i\partial_j u + \beta_i\partial_i u + \gamma u = f \qquad \text{in} \quad \Omega,$$

where a_{ij} satisfies the uniform ellipticity condition, and that we want to implement the boundary condition

$$\eta_i\partial_i u + \xi u = g \qquad \text{on} \quad \partial\Omega,$$

where all functions are assumed to be smooth on $\overline{\Omega}$. What conditions do we need to impose on η and ξ so that we can use the approach developed in this exercise? Try to find the weakest condition possible. (*Hint*: Consider the case $g = 0$ first.)

7. Consider the eigenvalue problem

$$Au = \lambda u,$$

on a bounded C^1 domain $\Omega \subset \mathbb{R}^n$, where A is as in one of the two preceding exercises, with $b_i = b_i' \equiv 0$. Prove the followings, by using the spectral theorem for compact self-adjoint positive operators where possible.

(a) The eigenvalues $\{\lambda_k\}$ are countable and real, and that $\lambda_k \to \infty$ as $k \to \infty$. Each eigenvalue has a finite multiplicity.

(b) The eigenfunctions $\{u_k\}$ form a complete orthonormal system in $L^2(\Omega)$.

(c) In the Dirichlet case, the system $\{u_k\}$ is complete and orthogonal in $H_0^1(\Omega)$, with respect to the inner product $a(u,v) + t\int_\Omega uv$, where t is a suitably chosen constant. The same holds for the Neumann case, with $H_0^1(\Omega)$ replaced by $H^1(\Omega)$.

(d) The eigenfunctions are smooth in Ω, and are smooth up to the boundary if $\partial\Omega$ is smooth. (Here you can use regularity results for elliptic equations without proof.)

(e) Explicitly compute the eigenvalues and eigenfunctions of the Laplacian with the homogeneous Dirichlet boundary condition on the rectangle $\Omega = (0,a) \times (0,b)$. Make sure that you don't miss any eigenfunctions, i.e., prove that under suitable scaling, the functions you computed form a complete orthonormal system in $L^2(\Omega)$.