MATH 580 ASSIGNMENT 1

DUE MONDAY SEPTEMBER 24

1. Let $\alpha > 0$, and let
$$f(x) = \begin{cases} e^{-(1-|x|^2)^{-\alpha}} & \text{for} \quad |x| < 1, \\ 0 & \text{for} \quad |x| \geq 1. \end{cases}$$
Prove that $f \in C^\infty(\mathbb{R}^n)$, but f is not real analytic. Exhibit a function $\phi \in C^\infty(\mathbb{R}^n)$, whose support is contained in $B_2 = \{x \in \mathbb{R}^n : |x| < 2\}$, such that $\phi \equiv 1$ on B_1.
2. Given a separating family $\mathscr{P}$ of seminorms on a vector space X, we say that a subset $A \subset X$ is *open* if for any $x \in A$, there exist finitely many seminorms $p_1, \ldots, p_k \in \mathscr{P}$, and a number $\varepsilon > 0$ such that $\{y \in X : \max_i p_i(y-x) < \varepsilon\} \subset A$.
   a) Verify that this notion satisfies the axioms of topology. That is, show that X is open, $\varnothing$ is open, intersection of any two open sets is open, and that the union of any collection of open sets is open.
   b) Show that the resulting space is *Hausdorff*, i.e., that for any $x, y \in X$ with $x \neq y$, there exist disjoint open sets $A \subset X$ and $B \subset X$ such that $x \in A$ and $y \in B$.
   c) Show that the resulting space is *locally convex*, i.e., that if $A \subset X$ is open and if $x \in A$ then there is a convex open set $C \subset A$ containing x.
3. Let Z be a topological space whose topology is induced by a separating family of seminorms, and let X and Y be normed spaces, both continuously embedded into Z. The latter means that $X \subset Z$ and $Y \subset Z$ as sets, and that the injections $x \mapsto x : X \to Z$ and $y \mapsto y : Y \to Z$ are continuous. Let $\{u_n\} \subset X \cap Y$ be a sequence such that $u_n \to x$ in X and $u_n \to y$ in Y. Show that $x = y$.
4. Let $\varphi \in \mathscr{D}(\mathbb{R})$, $\varphi \neq 0$, and $\varphi(0) = 0$. In each of the following cases, decide if $\varphi_j \to 0$ as $j \to \infty$ in $\mathscr{D}(\mathbb{R})$. Does it hold $\varphi_j \to 0$ pointwise or uniformly?
   a) $\varphi_j(x) = j^{-1}\varphi(x-j)$;
   b) $\varphi_j(x) = j^{-n}\varphi(jx)$, where $n > 0$ is an integer.
5. Show that in each of the following cases, f defines a distribution on $\mathbb{R}^2$, and find its order.
   a) $f(\varphi) = \int_{\mathbb{R}^2} |x|^{-1} e^{|x|^2} \varphi(x) \mathrm{d}x$;
   b) $f(\varphi) = \int_{\mathbb{R}} \varphi(s, 0) \mathrm{d}s$;
   c) $f(\varphi) = \int_0^1 \partial_1 \varphi(0, s) \mathrm{d}s$.
6. Compute the following derivatives in the sense of distributions.
   a) $\partial_x |x|$;
   b) $\partial_x \mathrm{sign}\, x$ ($\mathrm{sign}\, x = 0$ if $x = 0$ and $\mathrm{sign}\, x = x/|x|$ otherwise);

Date: Fall 2012.

c) $\partial_x \log |x|$;
d) $\partial_2 f$, where $f \in \mathscr{D}'(\mathbb{R}^2)$ is the distribution from b) of the previous exercise.

7. Prove the followings.
   a) $\partial_j(au) = (\partial_j a)u + a(\partial_j u)$ for $a \in C^\infty(\Omega)$ and $u \in \mathscr{D}'(\Omega)$.
   b) $\partial_j \partial_k u = \partial_k \partial_j u$ for $u \in \mathscr{D}'(\Omega)$.
   c) If $u_k \to u$ in $\mathscr{D}'(\Omega)$ then $\partial_j u_k \to \partial_j u$ in $\mathscr{D}'(\Omega)$.
   d) There is no distribution on $\mathbb{R}$ such that its restriction to $\mathbb{R} \setminus \{0\}$ is $e^{1/x}$.
8. Find the limits $n \to \infty$ of the following sequences in $\mathscr{D}'(\mathbb{R})$.
   a) $n\phi(nx)$, where ϕ is a nonnegative continuous function whose integral over $\mathbb{R}$ is finite.
   b) $\cos nx$.
   c) $n^k \sin nx$, where $k > 0$ is a constant.
   d) $x^{-1} \sin nx$.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.