The Yamabe Problem

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Abstract

The Yamabe problem asks if any Riemannian metric g on a compact smooth manifold M of dimension $n \geq 3$ is conformal to a metric with constant scalar curvature. The problem can be seen as that of generalizing the uniformization theorem to higher dimensions, since in dimension 2 scalar and Gaussian curvature are, up to a factor of 2, equal.

1 Introduction

In 1960, Yamabe claimed to have found a solution to the problem that would later come to bear his name. Eight years following, however, Neil Trudinger discovered a serious error in Yamabe's proof. Trudinger was able to salvage some of Yamabe's work but only by introducing further assumptions on the manifold M. In fact, Trudinger showed that there is a positive constant $\alpha(M)$ such that the result is true when $\lambda(M) < \alpha(M)$ ($\lambda(M)$ is the Yamabe invariant, to be defined later). In particular, if $\lambda(M) \leq 0$, the question is resolved. In 1976, Aubin improved on Trudinger's work by showing that $\alpha(M) = \lambda(S^n)$, where the n-sphere is equipped with its standard metric. Moreover, Aubin showed that if M has dimension $n \geq 6$ and is not locally conformally flat, then $\lambda(M) < \lambda(S^n)$. The remaining cases proved to be more difficult and it was not until 1984 that they had been resolved by Richard Schoen, thereby completing the solution to the Yamabe problem. In order to gain an appreciation for these developments and the final solution, it is necessary to look at Yamabe's approach. First we reiterate the statement of the problem.

The Yamabe problem: Let (M, g) be a C^∞ compact Riemannian manifold of dimension $n \geq 3$ and S its scalar curvature. Is there a metric $\tilde{g}$ conformal to g such that the scalar curvature $\tilde{S}$ of this new metric is constant?

2 Yamabe's approach

In this section we will detail Yamabe's attack on the problem. As we will see, he reduced the question of finding a conformal metric with constant scalar curvature to that of finding a smooth non-negative solution to a certain differential equation, the Yamabe equation. Because Krondrakov's theorem does not apply directly to the situation presented by that equation, Yamabe turned his attention to a family of approximating differential equations, which turn out to be the Euler-Lagrange equations of certain functionals. He proves smooth and non-negative solutions exist, and, claiming that these solutions are uniformly bounded and hence uniformly continuous, concludes that a solution Yamabe equation exists. An error had slipped into his argument however; the uniform boundedness is not true in general, and an explicit counter-example is found on the sphere.

2.1 The differential equation

Let M be a smooth, connected, and compact Riemannian manifold with metric g. We put $\tilde{g} = e^{2f}g$ with $f \in C^\infty(M)$. In a local chart, let Γ_{ij} and $\tilde{\Gamma}_{ij}$ denote the Christoffel symbols of g and $\tilde{g}$ respectively (throughout this section, given a symbol associated with (M, g), the same symbol with a tilde above denotes its counterpart in $(M, \tilde{g})$). Let R and S denote the Ricci and scalar curvatures of (M, g) respectively. Their difference is given by:

$$\Gamma_{ij} - \tilde{\Gamma}_{ij} = (g_{kl}\partial_i f + g_{ki}\partial_j f - g_{ij}\partial_k f)\, g^{kl}$$

The Ricci curvatures are then related in the following way:

$$\tilde{R}_{jk} = R_{jk} - (n-2)\nabla_{jk} f + (n-2)\nabla_j f \nabla_k f + (\nabla f - (n-2)|\nabla f|^2) g_{jk}$$

from which we derive:

$$\tilde{S} = e^{-2f}(S + 2(n-1)\nabla f - (n-1)(n-2)|\nabla f|^2)$$

where Δ is the Laplace-Beltrami operator. The above formula simplifies if we put $e^{2f} = \varphi^{p-2}$ with $p = 2n/(n-2)$ and $\tilde{g} = \varphi^{p-2}g$:

$$\tilde{S} = \varphi^{1-p}\left(4\frac{n-1}{n-2}\Delta\varphi + S\varphi\right) \tag{1}$$

We use the notation $a = 4\frac{n-1}{n-2}$ and $\Box = a\Delta + S$. The above implies that $\tilde{g} = \varphi^{p-2}g$ has constant scalar curvature iff it satisfies the *Yamabe equation*:

$$\Box\varphi = \lambda\varphi^{p-1} \tag{2}$$

This can be seen as a nonlinear eigenvalue problem. In fact, the equation $\Box\varphi = \lambda\varphi^q$ depends heavily on q. When q is close to 1, its behavior is quite similar to the that of the eigenvalue problem for $\Box$. If q is very large however, linear theory is not useful. It turns out that the exponent in the Yamabe equation is the critical value below which the equation is easy to solve and above which it may be unsolvable. In fact Yamabe had studied the more general equation

$$\Delta\varphi + h(x)\varphi = \lambda f(x)\varphi^{N-1} \tag{3}$$

where $N = p$, $h, f \in C^\infty(M)$, and $\varphi \not\equiv 0$ is a non-negative function in H_1 (the first Sobolev space). He wished to find $\lambda \in \mathbb{R}$ such that the above problem admits a solution. He considered the functional

$$I_q(\varphi) = \left[\int_M \nabla^i\varphi\nabla_i\varphi\; dV + \int_M h(x)\varphi^2\; dV\right]\left[\int_M f(x)\varphi^q\right]^{-2/q} \tag{4}$$

where $2 < q \leq N$. The denominator is defined since $H_1 \subseteq L_N \subseteq L_q$ (2.21). Yamabe next shows the above functional with $q = N$ has (3) for its Euler-Lagrange equation. Define

$$\mu_q = \inf\{I_q(\varphi) : \varphi \in H_1,\ \varphi \geq 0,\ \varphi \not\equiv 0\}$$

As we will soon see, showing μ_N (as $N = p$, this is the case we are most interested in) is attained directly is difficult since we cannot invoke Krondrakov's theorem where we would like. Thus Yamabe turned his attention to the approximate equations with $q < N$:

$$\Delta\varphi + h(x)\varphi = \lambda f(x)\varphi^{q-1} \tag{5}$$

Theorem (Yamabe): *For $2 < q < N$, there exists a C^∞ strictly positive φ_q satisfying* (5) *with $\lambda = \mu_q$ and $I_q(\varphi_q) = \mu_q$.*

Proof: The proof is split into several parts.

a) For $2 < q \leq N$, μ_q is finite. Indeed

$$I_q(\varphi) \geq \left[\inf_{x\in M}(0, h(x))\right]\left[\sup_{x\in M} f(x)\right]^{-2/q} ||\varphi||_2^2||\varphi||_q^{-2}$$

and

$$||\varphi||_2^2||\varphi||_q^{-2} \leq V^{1-2/q} \leq \sup(1, V^{2/n})$$

with $V = \int_M dV$. On the other hand,

$$\mu_q \leq I_q(1) = \left[\int_M h(x)dV\right]\left[\int_M f(x)dV\right]^{-2/q}$$

b) Let $\{\varphi_i\}$ be a minimizing sequence such that $\int_M f(x)\varphi_i^q dV = 1$:

$$\varphi_i \in H_1,\ \varphi_i \geq 0,\ \lim_{i\to\infty} I_q(\varphi_i) = \mu_q$$

First we prove that the set of the φ_i is bounded in H_1,

$$||\varphi_i||_{H_1}^2 = ||\nabla\varphi_i||_2^2 + ||\varphi_i||_2^2 = I_q(\varphi_i) - \int_M h(x)\varphi_i^2 dV + ||\varphi_i||_2^2$$

Since we can suppose that $I_q(\varphi_i) < \mu_q + 1$, then

$$||\varphi_i||_{H_1}^2 \leq \mu_q + 1 + \left[1 + \sup_{x\in M}|h(x)|\right]||\varphi_i||_2^2$$

and

$$||\varphi_i||_2^2 \le [V]^{1-2/q}||\varphi_i||_q^2 \le [V]^{1-2/q}\left[\inf_{x\in M} f(x)\right]^{-2/q}$$

c) If $2 < q < N$, there exists a non-negative function $\varphi_q \in H_1$ satisfying

$$I_q(\varphi_q) = \mu_q$$

and

$$\int_M f(x)\varphi_q^q dV = 1$$

Indeed for $2 < q < N$ the imbedding $H_1 \subseteq L_q$ is compact by Kondrakov's theorem (A2) and since the bounded closed sets in H_1 are weakly compact (A7), there exists a subsequence of $\{\varphi_i\}$, $\{\varphi_j\}$, and a function $\varphi_q \in H_1$ such that

1. $\varphi_j \to \varphi_q$ in L_q.
2. $\varphi_j \to \varphi_q$ weakly in H_1.
3. $\varphi_j \to \varphi_q$ a.e..

The last assertion is true by A8. The first assertion implies

$$\int_M f(x)\varphi_q^q dV = 1$$

and the third implies $\varphi_q \ge 0$. Finally the second implies

$$||\varphi_q||_{H_1} \le \liminf_{i\to\infty} ||\varphi_j||_{H_1}$$

Hence

$$I_q(\varphi_q) \le \lim_{j\to\infty} I_q(\varphi_j) = \mu_q$$

because $\varphi_j \to \varphi_q$ in L_2 according to the first assertion above, since $q \ge 2$. Therefore, by definition of μ_q, $I_q(\varphi_q) = \mu_q$.

d) φ_q satisfies (5) weakly in H_1. We compute the Euler-Lagrange equation. Set $\varphi = \varphi_q + \nu\psi$ with $\psi \in H_1$ and $\nu \in \mathbb{R}$ small. An asymptotic expansion gives:

$$\begin{aligned} I_q(\varphi) \quad &= \quad I_q(\varphi)\left[1 + \nu q \int_M f(x)\varphi_q^{q-1}\psi dV\right]^{-2/q} \\ &\quad +2\nu\left[\int_M \nabla^i\varphi_q\nabla_i\psi dV + \int_M h(x)\varphi_q\psi dV\right] + O(\nu) \end{aligned}$$

thus φ_q satisfies for all $\psi \in H_1$:

$$\int_M \nabla^i \varphi_q \nabla_i \psi dV + \int_M h(x)\varphi_q \psi dV = \mu_q \int_M f(x)\varphi_q^{q-1}\psi dV$$

which is the weak form of (5) with $\lambda = \mu_q$. To check that the preceding computation is correct, we note that since $\mathscr{D}(M)$ is dense in H_1 and $\varphi \not\equiv 0$, then

$$\inf_{\varphi\in H_1} I_q(\varphi) = \inf_{\varphi\in C^\infty} I_q(\varphi) = \inf_{\varphi\in C^\infty} I_q(|\varphi|) \geq \inf_{\varphi\in H_1,\varphi>0} I_q(\varphi) \geq \inf_{\varphi\in H_1} I_q(\varphi)$$

$I_q(\varphi) = I_q(|\varphi|)$ when $\varphi \in C^\infty$ because the set of points P where $\varphi(P) = 0$ and $|\nabla\varphi(P)| \neq 0$ is null.

e) $\varphi_q \in C^\infty$ for $2 \leq q < N$ and the functions φ_q are uniformly bounded for $2 \leq q \leq q_0 < N$. Let $G(P,Q)$ be the Green's function. φ_q satisfies the integral equation

$$\begin{aligned}\varphi_q(P) \quad &= \quad V^{-1}\int_M \varphi_q(Q)dV(Q) \\ &\quad + \int_M G(P,Q)\left[\mu_q f(Q)\varphi_q^{q-1} - h(Q)\varphi_q\right]dV(Q)\end{aligned} \tag{6}$$

We know that $\varphi_q \in L^{r_0}$ with $r_0 = N$. Since by A6 part 3 there exists a constant B such that $|G(P,Q)| \leq B[d(P,Q)]^{2-n}$, then according to Sobolev's lemma A1 and its corollary, $\varphi_q \in L^{r_1}$ for $2 < q \leq q_0$ with

$$\frac{1}{r_1} = \frac{n-2}{n} + \frac{q_0-1}{r+0} - 1 = \frac{q_0-1}{r_0} - \frac{2}{n}$$

and there exists a constant A_1 such that $||\varphi_q||_{r_1} \leq A_1||\varphi_q||_{r_0}^{q-1}$. By induction we see that $\varphi_q \in L^{r_k}$ with

$$\frac{1}{r_k} = \frac{q_0-1}{r_{k-1}} - \frac{2}{n} = \frac{(q_0-1)^k}{r_0} - \frac{2}{n}\frac{(q_0-1)^k - 1}{q_0-2}$$

and there exists a constant A_k such that $||\varphi_q||_{r_k} \leq A_k||\varphi_q||_{r_0}^{(q-1)^k}$. If for k large enough, $1/r_k$ is negative, then $\varphi_q \in L_\infty$. Indeed suppose $1/r_{k-1} > 0$ and $1/r_k < 0$. Then $(q_0 - 1)/r_{k-1} - 2/n < 0$ and Holder's inequality A4 applied to (6) yields $||\varphi_q||_\infty \leq C||\varphi_q||_{r_{k-1}}^{q-1}$ where C is a constant. There exists a k such that

$$\frac{1}{r_k} = (q_0-1)^k\left[\frac{1}{r_0} - \frac{2}{n(q_0-2)}\right] + \frac{2}{n(q_0-2)} < 0$$

because $n(q_0-2) < 2r_0 = 2N$, since $q_0 < N = \frac{2n}{n-2}$. Moreover, there exists a constant A_k which does not depend on $q \leq q_0$ such that

$$||\varphi_q||_\infty \leq A_k||\varphi_q||_N^{(q-1)^k}$$

But the set of the functions φ_q is bounded in H_1 (same proof as in part b). Thus by the Sobolev imbedding theorem A3, the functions φ_q are uniformly bounded. Since $\varphi_q \in L_\infty$, differentiating (6) yields $\varphi_q \in C^1$. φ_q satisfies (5); thus $\Delta\varphi_q$ is in C^1 and $\varphi_q \in C^2$ according to A5.

f) φ_q is strictly positive. This is true because

$$\int_M f(x)\varphi_q^q dV = 1$$

Then A9 establishes this result since φ_q cannot be identically 0. Lastly $\varphi_q \in C^\infty$ by induction according to A5. □

Remark: We cannot use the same method for the case $q = N$. This is because in c) we cannot apply Krondrakov's theorem and therefore only have

$$\int_M f(x)\varphi_N^N \leq 1$$

What's more, the method in e) gives us nothing as $q_0 = N$. In this case $r_k = r_0 = N$ for all k.

2.2 The Yamabe invariant

In view of the preceding theorem, to solve (2) Yamabe had considered the functional

$$J_q(\varphi) = \left[4\frac{n-1}{n-2}\int_M \nabla^i\varphi\nabla_i\varphi dV + \int_M S\varphi^2 dV\right]||\varphi_q||^{-2} \tag{7}$$

This is known as the *Yamabe functional.* which is the functional (4) corresponding to (5), divided by a constant. Define $\lambda_q(M) = \inf J_q(\varphi)$ for all $\varphi \geq 0$, $\varphi \not\equiv 0$, belonging to H_1. Set $\lambda(M) = \lambda_N(M)$ and $J(\varphi) = J_N(\varphi)$. In view of the following proposition, $\lambda(M)$ is appropriately called the *Yamabe invariant.* When M is understood, we will simply write λ and λ_q.

Proposition: *$\lambda(M)$ is a conformal invariant.*

Proof. Consider a change of conformal metric define by $\tilde{g} = \varphi^{\frac{4}{n-2}}g$. We have $dV' = \varphi^N dV$ and

$$J(\varphi\psi) = \frac{4\frac{n-1}{n-2}\left[\int_M \varphi^2\nabla^i\psi\nabla_i\psi dV + \int_M \varphi\psi^2\Delta\varphi dV\right] + \int_M S\varphi^2\psi^2 dV}{\left[\int_M \varphi^N\psi^N dV\right]^{2/N}}$$

using (1) gives $J(\varphi\psi) = \tilde{J}(\psi)$ and consequently $\lambda = \tilde{\lambda}$. □

2.3 Trudinger's work

From now on, we will assume without loss of generality that M has unit volume. Yamabe had claimed that the functions φ_q corresponding to J_q are uniformly bounded and therefore uniformly continuous, from which he showed a solution to (2) with the desired properties exists by a limiting argument. However, he was in error. The fault was discovered by

Trudinger in 1968. In the next section we will see a counter-example. The following theorem is due to Trudinger:

Theorem (Trudinger): *There exists* $\alpha(M) > 0$ *such that if* $\lambda(M) < \alpha(M)$*, then there exists a positive* C^∞ *solution to* (2) *with* $\tilde{S} = \lambda(M)$*. Thus Yamabe's problem is solved under this assumption on* (M, g)*.*

We do not prove this result as it is subsumed by the work of Aubin, which we shall look at next. We observe that if (M, g) has non-positive mean scalar curvature, then the above implies (M, g) indeed admits a conformal metric with constant scalar curvature; indeed one easily sees $J(1)$ is, up to a positive constant, the mean scalar curvature and since $\lambda(M) \leq J(1)$, the result immediately follows.

3 Results by Aubin

Aubin looked attentively at the Yamabe problem on the unit sphere. He had shown that $\lambda(S^n) = n(n-1)\omega_n^{2/n}$ where ω is the n-dimensional volume of S^n. Furthermore, he showed that the constant in the sharp Sobolev inequality is obtained from $\lambda(S^n)$. This allowed him to show that for any manifold M, $\lambda(M) \leq \lambda(S^n)$. If the inequality is strict, then the Yamabe problem is solved; thus $\alpha(M)$ above may be taken to be $\lambda(S^n)$.

3.1 Yamabe's problem on the sphere

Central to the treatment of Yamabe's problem is a firm grasp of the case of the sphere S^n. Here it is shown that the infimum for the functional (7) is attained by the standard metric on S^n which we will denote throughout this section by $\bar{g}$. The proof is based on an argument by Obata with simplifications by Penrose.

Theorem (Obata). *If* g *is a metric on* S^n *that is conformal to the standard metric* $\bar{g}$ *and has constant scalar curvature, then up to a constant scale factor,* g *is obtained from* $\bar{g}$ *by a conformal diffeomorphism of the sphere.*

Proof. We begin by showing g is Einstein (i.e. the Ricci tensor is proportional to g). From here on, the metric equipped to S^n is g. We can write $\overline{g} = \varphi^{-2} g$ where $\varphi \in C^\infty(S^n)$ is strictly positive. Making the substitution $e^{2f} = \varphi^{-2}$, we compute

$$\overline{R_{jk}} = R_{jk} + \varphi^{-1}\left((n-2)\nabla_{jk}\varphi - (n-1)\frac{\nabla_i\varphi\nabla^i\varphi}{\varphi}g_{jk} - \Delta\varphi g_{jk}\right)$$

in which the covariant derivatives and Laplacian are to be taken with respect to g, *not* with respect to $\overline{g}$. If $B_{jk} = R_{jk} - (S/n)g_{jk}$ represents the traceless Ricci tensor, then since $\overline{g}$ is Einstein,

$$0 = \overline{B}_{jk} = B_{jk} + (n-2)\varphi^{-1}(\nabla_{jk}\varphi + (1/n)\Delta\varphi g_{jk})$$

since the scalar curvature S is constant, the contracted Bianchi identity (B1) implies that divergence $R^i_{m,i}$o f the Ricci tensor vanishes identically, and thus so also does $B^i_{m,i}$. Because

B_{jk} is traceless, integration by parts gives

$$\begin{aligned}
\int_{S^n} \varphi |B|^2 dV_g &= \int_{S^n} \varphi B_{jk} B^{jk} dV_g \\
&= -(n-2) \int_{S^n} B^{jk} \left(\varphi_{jk} + \frac{1}{n} \Delta \varphi g_{jk} \right) dV_g \\
&= -(n-2) \int_{S^n} B^{jk} \varphi_{jk} dV_g \\
&= (n-2) \int_{S^n} B^{jk}_{,k} \varphi_j dV_g = 0
\end{aligned}$$

Thus B_{jk} must be Einstein. Since g is conformal to the standard metric $\overline{g}$ on the sphere, which is locally conformally flat, we have $W = 0$ as well as $B = 0$. This implies that g has constant curvature, and so (S^n, g) is isometric to a standard sphere. The isometry is the desired conformal diffeomorphism. □

Let $P = (0, \ldots, 1)$ be the north pole on $S^n \subseteq \mathbb{R}^{n+1}$. Stereographic projection $\sigma : S^n - \{P\} \to \mathbb{R}^n$ is defined by $\sigma(\zeta^1, \ldots, \zeta^n, \xi) = (x^1, \ldots, x^n)$ for $(\zeta, \xi) \in S^n - \{P\}$ where

$$x^j = \frac{\zeta^j}{1 - \xi}$$

We can verify that σ is a conformal diffeomorphism. If ds^2 is the Euclidean metric on $\mathbb{R}^n$, then under σ, $\overline{g}$ corresponds to

$$\rho * \overline{g} = 4(|x|^2 + 1)^{-2} ds^2$$

where ρ denotes σ^{-1}. This can be written as $4u_1^{p-2} ds^2$ where

$$u_1(x) = (|x|^2 + 1)^{(2-n)/2}$$

By means of stereographic projection, it is simple to write down conformal diffeomorphisms of the sphere the group of such diffeomorphisms is generated by the rotations, together with maps of the form $\sigma^{-1}\tau_\nu\sigma$ and $\sigma^{-1}\delta_\alpha\sigma$ where $\tau_\nu, \delta_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ are respectively translation by $\nu \in \mathbb{R}^n$:

$$\tau_\nu(x) = x - \nu$$

and dilation by $\alpha > 0$;

$$\delta_\alpha(x) = \alpha^{-1} x$$

The sphereical metric on $\mathbb{R}^n$ transforms under dilations to

$$\delta_\alpha^* \rho^* \overline{g} = 4u_\alpha^{p-2} ds^2$$

where $u_\alpha(x) = \left(\frac{|x|^2 + \alpha^2}{\alpha} \right)^{(2-n)/2}$.

Theorem. *There exists a positive C^∞ function ψ on S^n satisfying $J(\psi) = \lambda(S^n)$.*

Proof. For $2 \le q < p$, let φ_q be the solution on S^n to the subcritical problem (5). Composing with a rotation, we may assume that the supremum of φ_q is attained at the south pole for q. If $\{\varphi_q\}$ is uniformly bounded, then following the argument proposed by Yamabe we can show that a subsequence converges to an extremal solution, so assume $\sup \varphi_q \to \infty$. Now let $\kappa_\alpha = \sigma^{-1}\delta_\alpha\sigma : S^n \to S^n$ be the conformal diffeomorphism induced by dilation on $\mathbb{R}^n$ as described above. If we set $g_\alpha = \kappa_\alpha^* \overline{g}$, we can write $g_\alpha = t_\alpha^{p-2}\overline{g}$ where the conformal factor t_α is the function

$$t_\alpha(\zeta, \xi) = \left(\frac{(1+\xi) + \alpha^2(1-\xi)}{2\alpha} \right)^{(2-n)/2}$$

Observe that at the south pole $t_\alpha = \alpha^{(2-n)/2}$. For each $q < p$, let $\psi_q = t_\alpha \kappa_\alpha^* \varphi_q$, with $\alpha = \alpha_q$ chosen so that $\psi_q = 1$ at the south pole. This implies that $\alpha_q = (\sup \varphi_q)^{2/(n-2)} \to \infty$ as $q \to p$ and $\psi_q \le \alpha^{(n-2)/2} t_\alpha$ on M. Let $\Box_\alpha$ denote the conformally invariant Laplacian with respect to the metric g_α; by naturality of $\Box$, $\Box_\alpha(\kappa_\alpha^*\varphi_q) = \kappa_\alpha^*(\Box\varphi_q)$. Then by computation we find

$$\begin{aligned} \Box\psi_q &= \Box(t_\alpha \kappa_\alpha^* \varphi_q) = t_\alpha^{p-1} \Box_\alpha(\kappa_\alpha^*\varphi_q) = \lambda_q t_\alpha^{p-1}(\kappa_\alpha^*\varphi_q)^{q-1} \\ &= \lambda_q t_\alpha^{p-q} \psi_q^{q-1} \end{aligned} \tag{8}$$

where λ_q is defined as section 2.2. Observe that this transformation law also implies that

$$||\psi_q||_{2,1} \le C \int_{S^n} \psi_q \Box \psi_p dV_{\overline{g}} = C \int_{S^n} \varphi_q \Box \varphi_q dV_{\overline{g}} \le C' ||\varphi_q||_{2,1}$$

so $\{\psi_s\}$ is bounded in $L_1^2(S^n)$ and hence also in $L^p(S^n)$ by the Sobolev theorem. Let $\psi \in L_1^2(S^n)$ denote the weak limit.

Now if P is the north pole, on any compact subset of $S^n - \{P\}$ there exists a constant A such that $t_\alpha \le A\alpha^{(2-n)/2}$ and thus the RHS of (8) is bounded there by $\lambda_2 A^{p-1}$ independently of q. This implies that on any such set the RHS is bounded in L^r for every r. One can argue using local elliptic regularity that $\{\psi_q\}$ is bounded in $C^{2,\alpha}$ on compact sets disjoint from P. Let $K_1 \subseteq K_2 \subseteq \cdots$ be a sequence of compact sets whose union is $S^n - \{P\}$. By the Arzela-Ascoli theorem, we can choose a subsequence of $\{\psi_q\}$ that converges in $C^2(K_1)$ and then a subsequence that converges in $C^2(K_2)$, etc. Taking a diagonal subsequence, we see that the limit function ψ is C^2 on $S^n - \{P\}$.

Since $\lambda_p \to \lambda(S^n)$ and $t_\alpha^{p-q} \le 1$ away from P for q near p, we conclude that ψ satisfies $\Box\psi = f\psi^{p-1}$ on $S^n - \{P\}$ for some C^2 function f with $0 \le f \le \lambda(S^n)$. By the removable singularities result (A...) the same equation must hold weakly on all of S^n. For each p,

$$\begin{aligned} ||\psi_q||_p^p &= \int_{S^n} t_\alpha^p (\kappa_\alpha^* \varphi_q)^p dV_{\overline{g}} \\ &= \int_{S^n} (\kappa_\alpha^*\varphi_s)^p \kappa_\alpha^* dV_{\overline{g}} = ||\varphi_q||_p^p \ge \mathrm{Vol}(S^n)^{1-p/q} ||\varphi_q||_q^p \end{aligned}$$

This implies that $||\psi||_p \ge 1$ and therefore $J(\psi) \le \lambda(S^n)$. But since $\lambda(S^n)$ is by the definition the infimum of J, we must have $\Box\psi = \lambda(S^n)\psi^{p-1}$ and $J(\psi) = \lambda(S^n)$. It remains to show that ψ is positive and smooth. This can be accomplished by using the regularity theorem. □

The two results above combine to give us:

Theorem. *The Yamabe functional (7) on $(S^n, \bar{g})$ is minimized by constant multiples of the standard metric and its images under conformal diffeomorphisms. These are the only metrics conformal to the standard one on S^n that have constant scalar curvature.*

Another result by Aubin gives us $\lambda(S^n) = n(n-1)\omega_n^{2/n}$, where ω_n is the volume of S^n.

3.2 The sharp Sobolev inequality

Suppose $\frac{1}{r} = \frac{1}{q} - \frac{k}{n}$. Then $L_k^q(\mathbb{R}^n)$ is continuously embedded in $L^r(\mathbb{R}^n)$. In particular, for $q = 2$, $k = 1$, $r = p = \frac{2n}{n-2}$, we have the following Sobolev inequality:

$$||\varphi||_p^2 \leq \sigma_n \int_{\mathbb{R}^n} |\nabla\varphi|^2 dx,\ \ \varphi \in L_1^2(\mathbb{R}^n)$$

We call the smallest such constant σ_n the *best n-dimensional Sobolev constant* (on $\mathbb{R}^n$). The above results can be used to prove that in fact

$$\begin{aligned} \sigma_n &= a/\lambda(S^n) \\ &= 2(\omega_n)^{-1/n}[n(n-2)]^{-1/2} \end{aligned}$$

(one uses stereographic projection to convert the Yamabe problem on the sphere to an equivalent problem on $\mathbb{R}^n$ and uses the conformal invariance of J). Thus the sharp form of the Sobolev inequality is

$$||\varphi||_p^2 \leq \frac{a}{\lambda(S^n)} \int_{\mathbb{R}^n} |\nabla\varphi|^2$$

and we can show that equality is attained only by constant multiples and translates of the functions u_α defined above. The following result will also be useful to us:

Theorem. Let M be a compact Riemannian manifold with metric g, $p = 2n/(n-2)$, and let σ_n be the best Sobolev constant. Then for every $\varepsilon > 0$ there exists a constant C_ε such that for all $\varphi \in C^\infty(M)$,

$$||\varphi||_p^2 \leq (1+\varepsilon)\sigma_n \int_M |\nabla\varphi|^2 dV_g + C_\varepsilon \int_M \varphi^2 dV_g$$

3.3 Aubin's Theorems

We are in the position to prove:

Theorem (Aubin). If M is any compact Riemannian manifold of dimension $n \geq 3$, then $\lambda(M) \leq \lambda(S^n)$.

Proof. The functions u_α satisfy $a||\nabla u_\alpha||_2^2 = \lambda(S^n)||u_\alpha||^2$ on $\mathbb{R}^n$. For any fixed $\varepsilon > 0$, let B_ε denote the ball of radius ε in $\mathbb{R}^n$ and choose a smooth radial cutoff function $0 \leq \eta \leq 1$

supported in $B_{2\varepsilon}$ with $\eta \equiv 1$ on B_ε. Consider the smooth, compactly supported function $\varphi = \eta u_\alpha$. Since φ is a function of $r = |x|$ alone,

$$\begin{aligned}\int_{\mathbb{R}^n} a|\nabla\varphi|^2 dx &= \int_{B_{2\varepsilon}} (a\eta^2|\nabla u_\alpha|^2 + 2a\eta u_\alpha \langle \nabla\eta, \nabla u_\alpha\rangle + a u_\alpha^2|\nabla\eta|^2)dx \\ &\leq \int_{\mathbb{R}^n} a|\partial_r u_\alpha|^2 dx + C\int_{A_\varepsilon} (u_\alpha|\partial_r u_\alpha| + u_\alpha^2)dx\end{aligned} \tag{9}$$

where $A_\varepsilon$denotes the annulus $B_{2\varepsilon} - B_\varepsilon$. To estimate these terms we observe that

$$\partial_r u_\alpha = (2-n)r\alpha^{-1}\left(\frac{r^2+\alpha^2}{\alpha}\right)^{-n/2}$$

and so $u_\alpha \leq \alpha^{(n-2)/2}r^{2-n}$ and $|\partial_r u_\alpha| \leq (n-2)\alpha^{(n-2)/2}r^{1-n}$. Thus for fixed ε, the second term in (9) is $O(\alpha^{n-2})$ as $\alpha \to 0$. For the first term,

$$\begin{aligned}\int_{\mathbb{R}^n} a|\partial_r u_\alpha|^2 dx &= \lambda(S^n)\left(\int_{B_\varepsilon} u_\alpha^p dx + \int_{\mathbb{R}^n - B_\varepsilon} u_\alpha^p dx\right)^{2/p} \\ &\leq \lambda(S^n)\left(\int_{B_{2\varepsilon}} \varphi^p + \int_{\mathbb{R}^n - B_\varepsilon} \alpha^n r^{-2n}\right)^{2/p} \\ &= \lambda(S^n)\left(\int_{B_{2\varepsilon}} \varphi^p\right)^{2/p} + O(\alpha^n)\end{aligned}$$

Therefore the Sobolev quotient of φ on $\mathbb{R}^n$ is less than $\lambda(S^n) + C\alpha^{n-2}$. On a compact manifold M, let $\varphi = \eta u_\alpha$ in normal coordinates $\{x^i\}$ in a neighborhood of $P \in M$ extended by zero to a smooth function on M. Since φ is a radial function and $g^{rr} \equiv 1$ in normal coordinates, we have $|\nabla\varphi|^2 = |\partial_r\varphi|^2$ as before. The only corrections to the above estimate are introduced by the scalar curvature term and the difference between dV_g and dx. Since $dV_g = (1 + O(r))dx$ in normal coordinates, the previous calculation gives

$$\begin{aligned}E(\varphi) &= \int_{B_{2\varepsilon}} (a|\nabla\varphi|^2 + S\varphi^2)dV_g \\ &\leq (1 + C\varepsilon)\left(\lambda(S^n)||\varphi||_p^2 + C\alpha^{n-2} + C\int_0^{2\varepsilon}\int_{S_r} u_\alpha^2 r^{n-1} d\omega dr\right)\end{aligned}$$

One can then argue that the last term is bounded by a constant multiple of α. Thus choosing ε and then α small, we can arrange that

$$J(\varphi) \leq (1 + C\varepsilon)(\lambda(S^n) + C\alpha)$$

which proves $\lambda(M) \leq \lambda(S^n)$. $\square$

Theorem (Trudinger, Aubin). *Suppose $\lambda(M) < \lambda(S^n)$, and let φ_q be the collection of functions defined before. There are constants $q_0 < p$, $r > p$ and $C > 0$ such that $||\varphi_q||_r \leq C$ for all $q \geq q_0$.*

Proof. Let $\delta > 0$. Multiplying (5) by $\varphi_q^{1+2\delta}$ and integrating, we obtain

$$\int_M (a \langle d\varphi_q, (1+2\delta)\varphi_q^{2\delta} d\varphi_q \rangle + S\varphi_q^{2+2\delta}) dV_g = \lambda_q \int_M \varphi_q^{q+2\delta} dV_g$$

if we set $w = \varphi_q^{1+\delta}$ this can be written as

$$\frac{1+2\delta}{(1+\delta)^2} \int_M a|dw|^2 dV_g = \int_M (\lambda_q w^2 \varphi_q^{q-2} - Sw^2) dV_g$$

Now applying the sharp Sobolev inequality for any $\varepsilon > 0$,

$$\begin{aligned} ||w||_p^2 &\leq (1+\varepsilon)\frac{a}{\lambda(S^n)} \int_M |dw|^2 dV_g + C_\varepsilon \int_M w^2 dV_g \\ &\leq (1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)} \int_M \frac{\lambda_q}{\lambda(S^n)} w^2 \varphi_q^{q-2} dV_g + C_\varepsilon^{'} ||w||_2^2 \\ &\leq 1+\varepsilon)\frac{(1+\delta)^2}{(1+2\delta)} \frac{\lambda_q}{\lambda(S^n)} ||w||_p^2 ||\varphi_q||_{(q-2)n/2}^{q-2} + C_\varepsilon^{'} ||w||_2^2 \end{aligned}$$

by Holder's inequality. Since $q < p$, $(q-2)n/2 < q$ and thus by Holder's inequality again $||\varphi_q||_{(q-2)n/2} \leq ||\varphi_q||_q = 1$. Now if $0 \leq \lambda(M) < \lambda(S^n)$, then for some $q_0 < p$, $\lambda_q/\lambda(S^n) \leq \lambda_{q_0} < 1$ for $q \geq q_0$. Thus we can choose ε and δ small enough so that the coefficient of the first term above is less than 1, and so can be absorbed in the LHS. Thus

$$||w||_p^2 \leq C||w||_2^2$$

The same result obviously holds if $\lambda(M)$ and hence λ_q is less than 0. But applying Holder's inequality once more we see that

$$||w||_2 = ||\varphi_q||_{2(1+\delta)}^{1+\delta} \leq ||\varphi_q||_q^{1+\delta} = 1$$

Therefore $||w||_p = ||\varphi_q||_{p(1+\delta)}^{1+\delta}$ is bounded independently of q. □

We can now prove the Yamabe problem has a solution if $\lambda(M) < \lambda(S^n)$. We first state a regularity theorem without giving proof (see theorem A9).

Theorem (Regularity). *Suppose $\varphi \in L_1^2(M)$ is a non-negative weak solution of* (5) *with $2 \leq q \leq p$ and $|\lambda_q| \leq K$ for some constant K. If $\varphi \in L^r(M)$ for some $r > (q-2)n/2$ (in particular if $r = s < p$ or if $s = p < r$) then φ is either identically* 0 *or strictly positive and C^∞ and $||\varphi||_{C^{2,\alpha}} \leq C$ where C depends only on M, g, K and $||\varphi||_r$.*

Theorem. *Let $\{\varphi_q\}$ be as before and assume $\lambda(M) < \lambda(S^n)$. As $q \to p$, a subsequence converges uniformly to a positive function $\varphi \in C^\infty(M)$ which satisfies $J(\varphi) = \lambda(M)$ and equation* (2).

Proof. Since the functions $\{\varphi_q\}$ are uniformly bounded in $L^r(M)$, the above regularity result shows that they are uniformly bounded in $C^{2,\alpha}(M)$ as well. The Arzela-Ascoli theorem then implies that a subsequence converges in the C^2 norm to a function $\varphi \in C^2(M)$. The limit function φ therefore satisfies

$$\Box\varphi = \lambda\varphi^{p-1},\ J(\varphi) = \lambda$$

where $\lambda = \lim_{q\to p}\lambda_q$. If $\lambda(M) \geq 0$, then one can show that $\lambda = \lambda(M)$. On the other hand, if $\lambda(M) < 0$, the fact that $\lambda_p$is increasing implies that $\lambda \leq \lambda(M)$ but since $\lambda(M)$ is the infimum of J, we must have $\lambda = \lambda(M)$ in this case as well. Another application of the regularity result in elliptic theory shows that φ is C^∞ and is strictly positive because $||\varphi||_p \geq \lim_{s\to p}||\varphi_q||_q = 1$. $\Box$

The above shows that Yamabe's problem is resolved in the affirmative provided $\lambda(M) < \lambda(S^n)$. It is normal to ask then if this is true for all manifolds M; the Yamabe problem would then be completely resolved. In the case $n \geq 6$ and M is not locally conformally flat, this turns out to be true.

Definition: A Riemannian manifold M is said to be *locally conformally flat* if each point has a neighborhood where there exists a conformal metric whose curvature vanishes.

Theorem (Aubin). *If M has dimension n and $n \geq 6$ is a compact nonlocally conformally flat Riemannian manifold, then $\lambda(M) < \lambda(S^n)$. Hence the minimum $\lambda(M)$ is attained and equation* (1) *has a strictly positive solution with $\tilde{S} = \lambda(M)$, so Yamabe's problem is solved in this case.*

Proof. When $n \geq 4$, a necessary and sufficient condition for a manifold to be locally conformally flat is that the Weyl tensor vanishes identically. By definition the Weyl tensor is

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ &\quad + \frac{S}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{jk}g_{il}) \end{aligned}$$

and W^j_{ikl} is a conformal invariant. If M is not locally conformally flat there exists a point P where the Weyl tensor is not zero. After a change of conformal metric, the calculation of an asymptotic expansion yields a second term whose sign is $-W_{ikjl}W^{ijkl}$ when $n \geq 6$, thus showing $\lambda(M) < \lambda(S^n)$. $\Box$

4 Further results by Schoen

Richard Schoen proved the following theorem:

Theorem (Schoen): If M has dimension $3, 4$ or 5, or if M is locally conformally flat, then $\lambda(M) < \lambda(S^n)$ unless M is conformal to the standard sphere.

Schoen made heavy use of the Green function for the operator $\Box$. He also serendipitously found the positive mass theorem of general relativity, which had recently been proved in dimension 3 and 4 by Schoen and Yau. The proof of the above theorem requires an n-dimensional version of the positive mass theorem. The 5-dimensional case is a generalizing of the proof of the 4-dimensional case. The n-dimensional case, $n \geq 6$, is more difficult. However, by Aubin's results only conformally flat manifolds need be considered.

5 Appendix: Preliminary Results

We collect here a number of results useful to us. The proofs can be found in [Aub].

5.1 Analytic results

Theorem A1. (Sobolev's lemma). *Let $p' > 1$ and $q' > 1$ be real numbers. Define λ by $1/p' + 1/q' + \lambda/n = 2$. If λ satisfies $0 < \lambda < n$, there exists a constant $K(p', q', n)$ such that for all $f \in L^{q'}(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$:*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{||x-y||^\lambda} dxdy \leq K(p', q', n)||f||_{q'}||g||_{p'}$$

$||x||$ being the Euclidean norm.

Corollary. *Let λbe a real number $0 < \lambda < n$ and $q' > 1$. If r defined by $1/r = \lambda/n + 1/q' - 1$ satisfies $r > 1$ then*

$$h(y) = \int_{\mathbb{R}^n} \frac{f(x)}{||x-y||^\lambda} dx$$

belongs to L_r when $f \in L^{q'}(\mathbb{R}^n)$. Moreover there exists a constant $C(\lambda, q', n($ such that for all $f \in L_{q'}(\mathbb{R}^n)$

$$||h||_r \leq C(\lambda, q', n)||f||_{q'}$$

Theorem A2. (Kondrakov): *Let $k \geq 0$ be an integer and $p, q \in \mathbb{R}$ satisfying $1 \geq 1/p > 1/q - k/n > 0$. The Kondrakov theorem asserts that, if $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary then*

1. *The imbedding $W^{k,q}(\Omega) \subseteq L^p(\Omega)$ is compact.*
2. *The imbedding $W^{k,q}(\Omega) \subseteq C^\alpha(\overline{\Omega})$ is compact, if $k - \alpha > n/q$ with $0 \leq \alpha < 1$.*
3. *The following embeddings $W_0^{k,q} \subseteq L^p(\Omega)$ and $W_0^{k,q}(\Omega) \subseteq C^\alpha(\overline{\Omega})$ are compact.*

The Kondrakov theorem also holds for compact Riemannian manifold.

Definition: *The injectivity radius at a point P of a Riemannian manifold is the largest radius for which the exponential map at P is a diffeomorphism. The injectivity radius of a Riemannian manifold is the infimum of the injectivity radii at all points.*

Theorem A3. *The Sobolev imbedding theorem holds for M a complete manifold with bounded curvature and injectivity radius $\delta > 0$. Moreover for any $\varepsilon > 0$, there exists a constant $A_q(\varepsilon)$ such that every $\varphi \in H_1^q(M)$ satisfies*

$$||\varphi||_p \leq [K(n, q) + \varepsilon]||\nabla\varphi||_q + A_q(\varepsilon)||\varphi||_q$$

with $1/p = 1/q - 1/n > 0$ (see section 3.2).

Proposition A4 (Holder's inequality). *Let M be a Riemannian manifold. If $f \in L^r(M) \cap L^q(M)$, $1 \le r < q \le \infty$, then $f \in L^p$ for $p \in [r,q]$ and*

$$||f||_p \le ||f||_r^a ||f||_q^{1-\alpha}$$

with $a = \frac{1/p - 1/q}{1/r - 1/q}$.

Theorem A5. *Let Ω be an open set of $\mathbb{R}^n$ and $A = a_\ell \nabla^\ell$ a linear elliptic operator of order $2m$ with C^∞ coefficients ($a_\ell \in C^\infty(\Omega)$ for $0 \le \ell \le 2m$). Suppose $\mathfrak{u}$ is a distribution solution of the equation $A(\mathfrak{u}) = f$ and $f \in C^{k,\alpha}(\Omega)$ (resp. $C^\infty(\Omega)$). Then $\mathfrak{u} \in C^{k+2m,\alpha}(\Omega)$ (resp. $C^\infty(\Omega)$) with $0 < \alpha < 1$. If f belongs to $W^{k,p}(\Omega)$, $1 < p < \infty$ then $\mathfrak{u}$ belongs locally to $W^{k+2m,p}$.*

Theorem A6. *Let M be a compact C^∞ Riemannian manifold. There exists $G(P,Q)$ a Green's function of the Laplacian which has the following properties:*

1. *For all functions $\varphi \in C^2$:*

$$\varphi(P) = V^{-1} \int_M \varphi(Q) dV(Q) + \int_M G(P,Q) \Delta\varphi(Q) dV(Q)$$

2. *$G(P,Q)$ is C^∞ on $M \times M$ minus the diagonal (for $P \ne Q$).*
3. *There exists a constant k such that:*

$$\begin{aligned} |G(P,Q)| &< k(1 + |\log r|) \text{ for } n = 2 \text{ and} \\ |G(P,Q)| &< kr^{2-n} \text{ for } n > 2,\ |\nabla_Q G(P,Q)| < kr^{1-n} \\ |\nabla_Q^2 G(P,Q)| &< kr^{-n} \text{ with } r = d(P,Q) \end{aligned}$$

4. *There exists a constant A such that $G(P,Q) \ge A$. Because the Green function is defined up to a constant, we can thus choose the Green's function everywhere positive.*
5. *$\int G(P,Q) dV(P)$ is constant. We can choose the Green's function so that its integral is equal to 0.*
6. *$G(P,Q) = G(Q,P)$.*

Theorem A7. *A Banach space is reflexive iff the closed unit ball is weakly sequentially compact.*

Proposition A8. *Let $\{f_k\}$ be a sequence in L^p (or in L^∞) which converges in L^p to $f \in L^p$. Then there exists a subsequence converging pointwise to f a.e.*

Theorem A9. *Let M be a compact Riemannian manifold. If a function $\psi \ge 0$ belonging to $C^2(M)$ satisfies an inequality of the type $\Delta\psi \ge \psi f(P,\psi)$, where $f(P,t)$ is a continuous numerical function on $M \times \mathbb{R}$, then either ψ is strictly positive or ψ is identically 0.*

5.2 Geometric results

Theorem B1. (Bianchi identities): *Let R be the curvature tensor of a Riemannian manifold M. It satisfies the Bianchi identities: $R_{ijkl} + R_{iklj} + R_{iljk} = 0$, $R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0$.*

Theorem B2. Let (M, g) be a Riemannian metric and $\tilde{g} = e^{2f} g$ a conformal metric. Then we have the following transformation law for the Ricci curvature:

$$\tilde{R}_{jk} = R_{jk} - (n-2)\nabla_{jk} f + (n-2)\nabla_j f \nabla_k f + (\nabla f - (n-2)|\nabla f|^2) g_{jk}$$

where R and $\tilde{R}$ are the Ricci curvatures of g and $\tilde{g}$ respectively.

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