Symmetry analysis of a system of modified shallow-water equations

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We reconsider the symmetry analysis of a modified system of one-dimensional shallow-water equations (MSWE) recently studied by T. Raja Sekhar and V.D. Sharma [*Commun. Nonlinear Sci. Numer. Simulat.* **20** (2012) 630–636]. Only a finite dimensional Lie-subalgebra of the MSWE was found in the aforementioned paper, in fact that subalgebra is infinite dimensional. We show that the MSWE can be linearized using a hodograph transformation. An optimal list of inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra is constructed and used for Lie reduction. Non-Lie solutions are found by extending suitable solutions of the linearized modified shallow-water equations to solutions of the original system.

1 Introduction

The computation of exact solutions of nonlinear systems of partial differential equations remains an important task, despite the increased interest in the numerical simulations of such differential equations. Exact solutions are crucial, for instance, in the testing of numerical methods as well as in the analytical study of the associated partial differential equations.

Finding exact solutions remains rather challenging, unless the equations of interest belong to a category for which well-known solving methods exist. However, in general, the most we can do is to obtain particular solutions. Particular solutions, albeit often simple, can be of great help to gain understanding and insight into the principal dynamics of such systems of equations.

One way to obtain exact solutions of differential equations is through the study of their Lie symmetries (see e.g. [?, 10, 22] for a discussion on non-Lie methods for finding exact solutions). Lie symmetries and related computational methods for group-invariant solutions of system of differential equations is a well investigated subject. The current paper concerns itself with a hydrodynamical problem. Relevant results on symmetries and exact solutions can be found e.g. in the textbooks [1, 2, 9, 19, 22, 24, 25] and in the papers [3, 12, 14–16, 20, 21, 27].

The recent paper [30] intended to find Lie symmetries and exact solutions of the following system of one-dimensional shallow-water equations:

$$\Delta_1 := u_t + uu_x + g\left(1 + \frac{H}{h}\right)h_x = 0, \quad \Delta_2 := h_t + uh_x + hu_x = 0, \tag{1}$$

where u is the fluid velocity in x-direction, h is the height of the water column, and g and H are constants related to the gravity acceleration and momentum transport, respectively. The constant g can be scaled to $g = 1$ by an equivalence transformation. In the case of $H = 0$, this system reduces to the usual form of the one-dimensional shallow-water equations.

We found a number of incorrect and incomplete results in [30], which is why we reconsider the symmetry analysis of the system (1). Our main concern was that in (1) some of the invariant "solutions" that were found were not solutions. Moreover, the symmetry algebra is supposed to be infinite dimensional although in [30] it was found to be finite dimensional.

This sits ill with the fact that the system (1) can be linearized via a hodograph transformation and thus should display an infinite dimensional maximal Lie invariance algebra.

Another problem with the Lie reductions carried out in (1) is that no optimal list of inequivalent subalgebras was determined, which should be the base for an efficient computation of group-invariant solutions. As a consequence, the exact solutions found in [30] are in an overly complicated form.

The structure of this paper is as follows:

In section 2 we compute the Lie symmetries admitted by system (1). We find an infinite dimensional maximal Lie invariance algebra and compare it to the algebra found in [30] which is only finite dimensional. We indicate how the modified shallow-water equations are linearized through a hodograph transformation.

In section 3 we classify inequivalent one-dimensional subalgebras of the infinite dimensional maximal Lie invariance algebra of the modified shallow-water equations. The classification of inequivalent one-dimensional subalgebras of the symmetry algebra found in [30] is also given.

In section 4 we construct the group-invariant solutions based on the optimal system found in Section 3. We also determine non-Lie solutions of the modified shallow-water equations through solutions of the linearized modified shallow-water equations.

The results obtained are summed up in the final Section 5.

2 Lie symmetries of the system of modified shallow-water equations

In this section, we compute the maximal Lie invariance algebra $\mathfrak{g}$ of the system (1). This is done by determining the coefficients of the general infinitesimal generator

$$Q = \tau(t,x,u,h)\partial_t + \xi(t,x,u,h)\partial_x + \eta(t,x,u,h)\partial_u + \phi(t,x,u,h)\partial_h,$$

of a one-parameter Lie symmetry group. These coefficients are found using the infinitesimal invariance criterion, which in the present case reads

$$Q^{(1)}\Delta_1 = 0, \quad Q^{(1)}\Delta_2 = 0, \tag{2}$$

where this equality has to hold on the manifold defined by $\Delta_1 = 0$ and $\Delta_2 = 0$. In this last system, $Q^{(1)}$ denotes the first prolongation of the operator Q, which is of the form

$$Q^{(1)} = Q + \eta^t \partial_{u_t} + \eta^x \partial_{u_x} + \phi^t \partial_{h_t} + \phi^x \partial_{h_x}$$

where the coefficients are given by

$$
\begin{aligned}
\eta^t &= \mathrm{D}_t(\eta - \tau u_t - \xi u_x) + \tau u_{tt} + \xi u_{tx}, \quad \eta^x = \mathrm{D}_x(\eta - \tau u_t - \xi u_x) + \tau u_{tx} + \xi u_{xx}, \\
\phi^t &= \mathrm{D}_t(\phi - \tau h_t - \xi h_x) + \tau h_{tt} + \xi h_{tx}, \quad \phi^x = \mathrm{D}_x(\phi - \tau h_t - \xi h_x) + \tau h_{tx} + \xi h_{xx},
\end{aligned}
\tag{3}
$$

they follow from the *general prolongation formula*, see e.g. [2, 9, 10, 24, 25]. The total derivative operators D_t and D_x arising in the expression of the coefficients η^t, η^x, ϕ^t and ϕ^x of the prolonged operator $Q^{(1)}$ are given by

$$
\begin{aligned}
\mathrm{D}_t &= \partial_t + u_t\partial_u + h_t\partial_h + u_{tt}\partial_{u_t} + h_{tt}\partial_{h_t} + u_{tx}\partial_{u_x} + h_{tx}\partial_{h_x} + \cdots, \\
\mathrm{D}_x &= \partial_x + u_x\partial_u + h_x\partial_h + u_{tx}\partial_{u_t} + h_{tx}\partial_{h_t} + u_{xx}\partial_{u_x} + h_{xx}\partial_{h_x} + \cdots.
\end{aligned}
$$

Explicitly, the infinitesimal invariance criterion (2) reads

$$\eta^t + u\eta^x + \eta u_x + \left(1 + \frac{H}{h}\right)\phi^x - \frac{H}{h^2}\phi h_x = 0,$$

$$\phi^t + u\phi^x + \eta h_x + h\eta^x + \phi u_x = 0.$$

Plugging the coefficients (3) into the above system, substituting $u_t = -(uu_x + (1 + H/h)h_x)$ and $h_t = -(uh_x + hu_x)$ wherever they arise and splitting the resulting equations with respect to powers of the derivatives of u and h, we derive the following system of determining equations for coefficients of the vector field Q,

$$\xi_u - u\tau_u + h\tau_h = 0,$$

$$\xi_h - u\tau_h + \left(1 + \frac{H}{h}\right)\tau_u = 0,$$

$$\frac{H}{h^2}\phi - \left(1 + \frac{H}{h}\right)(\tau_t - \xi_x - \eta_u + \phi_h + 2u\tau_x) = 0,$$

$$\phi + h(\tau_t - \xi_x + \eta_u - \phi_h + 2u\tau_x) = 0,$$

$$\eta - h\eta_h + u(\tau_t - \xi_x) - \xi_t + u^2\tau_x + \left(1 + \frac{H}{h}\right)(\phi_u + h\tau_x) = 0,$$ (4)

$$\eta + h\eta_h + u(\tau_t - \xi_x) - \xi_t + u^2\tau_x - \left(1 + \frac{H}{h}\right)(\phi_u - h\tau_x) = 0,$$

$$\eta_t + u\eta_x + \left(1 + \frac{H}{h}\right)\phi_x = 0,$$

$$\phi_t + u\phi_x + h\eta_x = 0.$$

This system has two inequivalent solutions, depending on whether $H \neq 0$ or $H = 0$.

If $H \neq 0$, the system of determining equations (4) yields the solution

$$\tau = c_1 t + f(u, h), \quad \xi = c_1 x + c_2 t + g(u, h), \quad \eta = c_2, \quad \phi = 0,$$

where $c_1, c_2 \in \mathbb{R}$ and the functions f and g run through the set of solutions of the system

$$g_u - uf_u + hf_h = 0, \quad g_h - uf_h + \left(1 + \frac{H}{h}\right)f_u = 0.$$ (5)

If $H = 0$, system (1) reduces to the usual one-dimensional shallow-water equations for which the solution of the determining equations (4) is

$$\tau = c_1 t + c_4(2x - 6tu) + f(u, h), \quad \xi = c_1 x + c_2 t + c_3 x + c_4 t(6h - 3u^2) + g(u, h),$$
$$\eta = c_2 + c_3 u + c_4(u^2 + 4h), \quad \phi = 2c_3 h + 4c_4 uh,$$

where $c_1, \ldots, c_4 \in \mathbb{R}$ and the functions f and g run through the set of solutions of the system (5) in which $H = 0$. They give rise to the vector fields

$$\begin{aligned}
\mathcal{C}_1 &= t\partial_x + \partial_u \\
\mathcal{C}_2 &= t\partial_t + x\partial_x \\
\mathcal{C}_3 &= 2h\partial_h + u\partial_u - t\partial_t \\
\mathcal{C}_4 &= 4hu\partial_h + (4hg + u^2)\partial_u + (-6ut + 2x)\partial_t + (6hgt - 3u^2t)\partial_x \\
\mathcal{L}(f, g) &= f(u, h)\partial_t + g(u, h)\partial_x.
\end{aligned}$$

This case is well investigated [references] and so we will concentrate on $H \neq 0$.

3

When $H \neq 0$ one subsequently finds that the maximal Lie invariance algebra $\mathfrak{g}$ of system (1) is infinite dimensional and spanned by the vector fields

$$\mathcal{D} = t\partial_t + x\partial_x, \quad \mathcal{G} = t\partial_x + \partial_u, \quad \mathcal{L}(f,g) = f(u,h)\partial_t + g(u,h)\partial_x. \tag{6}$$

The first operator is associated with scaling symmetries in t and x and the second operator gives rise to Galilean boosts. Similarly to the usual one-dimensional shallow-water equations [7, 18], the operator $\mathcal{L}(f,g)$ indicates that system (1) can be linearized to system (5) by the hodograph transformation in which $f = t$ and $g = x$ serve as the new unknown functions and u and h become the new independent variables.

We have verified our computations, leading to the maximal Lie invariance algebra $\mathfrak{g}$, using the package `desolv` for finding Lie symmetries [31].

Besides Lie symmetries, also discrete point symmetries of systems of differential equations can be useful, especially in the classification of optimal lists of inequivalent subalgebras or the construction of structure-preserving low-dimensional dynamical systems, see e.g. [5, 8]. Discrete point symmetries can be computed systematically using the direct method or, often more convenient, a version of the algebraic method proposed in [17] and refined in [6, 11] for systems admitting infinite dimensional maximal Lie invariance algebras. In the present case, it is found that system (1) admits two independent point symmetries (up to composition with continuous symmetries), which are given by the pairwise alternation of signs in t, x and x, u, respectively.

Remark 1. In [30] the authors stated that the maximal Lie invariance algebra of system (1) is generated by the four vector fields

$$\partial_t, \quad \partial_x, \quad t\partial_x + \partial_u, \quad t\partial_t + x\partial_x.$$

Comparing this algebra, which we denote by $\mathfrak{g}^1$, with the algebra (6), it is obvious that $\mathfrak{g}^1$ is only a subalgebra of $\mathfrak{g}$, the first two operators being special instances of the generator $\mathcal{L}(f,g)$, for which $f = \mathrm{const}$, $g = 0$ and $f = 0$, $g = \mathrm{const}$, respectively. It is thus also missed in [30] that system (1) can be linearized using a hodograph transformation.

In the following, we denote by G^1 the subgroup of the maximal Lie invariance pseudogroup G of the modified shallow-water equations (1) that is associated with the subalgebra $\mathfrak{g}^1$ in [30].

3 Classification of subalgebras of the system of modified shallow-water equations

The computation of exact solutions of system (1) was of central importance in [30]. This was done with the method of group-invariant (or Lie) reduction. Unfortunately these reductions were constructed without a classification of optimal lists of inequivalent subalgebras, which is a common mistake. This topic is extensively discussed in e.g. [24, 25], where it is pointed out that only group-invariant solutions stemming from inequivalent subalgebras of the maximal Lie invariance algebra of a system of differential equations are guaranteed to be inequivalent. This means that they cannot be related to each other by means of a symmetry transformation.

Hence, the first step in the construction of exact solutions of a system of differential equations using Lie reduction should be the classification of subalgebras of appropriate dimensions. In the present case, system (1) is a system in $(1+1)$-dimensions, so it is sufficient to carry out the Lie reductions with respect to one-dimensional subalgebras of $\mathfrak{g}$, which then yield systems of ordinary differential equations. In the following, we classify both subalgebras of the infinite dimensional maximal Lie invariance algebra $\mathfrak{g}$ and the subalgebra $\mathfrak{g}^1$ found in [30], as no classification of one-dimensional subalgebras of $\mathfrak{g}^1$ was given in this paper.

The commutation relations between elements of $\mathfrak{g}$ are

$$[\mathcal{D}, \mathcal{G}] = 0, \quad [\mathcal{L}(f,g), \mathcal{D}] = \mathcal{L}(f,g), \quad [\mathcal{G}, \mathcal{L}(f,g)] = \mathcal{L}(f_u, g_u - f), \quad [\mathcal{L}(f_1, g_1), \mathcal{L}(f_2, g_2)] = 0,$$

where the functions f_1, g_1 and f_2, g_2 are solutions of system (5). The four nonidentical adjoint actions of the maximal Lie invariance pseudogroup G on the generating vector fields $\mathcal{D}$, $\mathcal{G}$ and $\mathcal{L}(f,g)$ then are

$$\mathrm{Ad}(e^{\varepsilon\mathcal{D}})\mathcal{L}(f,g) = e^{\varepsilon}\mathcal{L}(f,g), \quad \mathrm{Ad}(e^{\varepsilon\mathcal{L}(f,g)})\mathcal{D} = \mathcal{D} - \varepsilon\mathcal{L}(f,g),$$

$$\mathrm{Ad}(e^{\varepsilon\mathcal{G}})\mathcal{L}(f,g) = \mathcal{L}(f',g'), \quad \mathrm{Ad}(e^{\varepsilon\mathcal{L}(f,g)})\mathcal{G} = \mathcal{G} + \varepsilon\mathcal{L}(f_u, g_u - f),$$

where $f' = f(u - \varepsilon, h)$ and $g' = (g(u - \varepsilon, h) + \varepsilon f(u - \varepsilon, h))$. For each pair of generating vector fields $(\mathbf{v}, \mathbf{w}_0)$ of $\mathfrak{g}$, these adjoint actions can be computed either through the Lie series [24],

$$\mathbf{w}(\varepsilon) = \mathrm{Ad}(e^{\varepsilon\mathbf{v}})\mathbf{w}_0 := \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(\mathrm{ad}\ \mathbf{v})^n\mathbf{w}_0,$$

or through a pushforward of the vector fields from $\mathfrak{g}$ by elementary symmetry transformations from the maximal Lie invariance pseudogroup G, see [4,13] for more details. The second method is particularly suitable for computations involving infinite dimensional Lie algebras.

Now that we have the nonidentical adjoint actions at hand, we can proceed with the classification of the one-dimensional subalgebras of $\mathfrak{g}$. To accomplish this classification, we start with the most general element of a one-dimensional subalgebra from $\mathfrak{g}$,

$$\mathbf{v} = a\mathcal{D} + b\mathcal{G} + \mathcal{L}(f,g),$$

where $a, b \in \mathbb{R}$ and f and g are arbitrary but fixed functions of the system (5). We subsequently simplify it as much as possible by applying the adjoint actions that we found previously. As the classification is rather short, we will give it explicitly here.

(i) If $a \neq 0$, we can scale $a = 1$ and use the adjoint action $\mathrm{Ad}(e^{\varepsilon\mathcal{L}(\tilde{f},\tilde{g})})$ to cancel $\mathcal{L}(f,g)$ from $\mathbf{v}$. No further simplifications are possible and so the simplified form of $\mathbf{v}$ is $\mathbf{v} = \mathcal{D} + b\mathcal{G}$.

(ii) For $a = 0$ and $b \neq 0$, we can set $b = 1$ and the adjoint action $\mathrm{Ad}(e^{\varepsilon\mathcal{L}(\tilde{f},\tilde{g})})$ can be used again to cancel $\mathcal{L}(f,g)$, and thus $\mathbf{v} = \mathcal{G}$ in this case.

(iii) The final one-dimensional subalgebra is spanned by $\mathbf{v} = \mathcal{L}(f,g)$.

Collecting the results obtained, we have thus proved the following theorem.

Theorem 1. *The optimal list of inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra $\mathfrak{g}$ of the system of modified shallow-water equations* (1) *consists of the subalgebras*

$$\langle \mathcal{D} + a\mathcal{G}\rangle, \quad \langle\mathcal{G}\rangle, \quad \langle\mathcal{L}(f,g)\rangle,$$

where $a \in \mathbb{R}$ and f and g are arbitrary but fixed functions that are solutions of the system (5).

In principle, Theorem 1 tells us which subalgebras should be used for the Lie reductions of system (1). At the same time it is useful to classify the subalgebra $\mathfrak{g}^1$ found in [30], for which no optimal system of one-dimensional subalgebras was given. This is a simple task, since the classification of subalgebras of real four-dimensional Lie algebras has already been discussed in the literature. See [23, 26, 28] for classifications of real low-dimensional algebras of dimensions up to six. For Lie algebras of dimension higher than six exhaustive classification results exist, provided that the Lie algebra possesses a special structure: see the review in [28] for more details.

In the present case, the algebra $\mathfrak{g}^1$ is isomorphic to the algebra $A_{4,8}^0$ in the classification [28], the nilradical of which is $\langle\partial_t, \partial_x, \mathcal{G}\rangle$, which is isomorphic to the Heisenberg algebra. The one-dimensional subalgebras of this algebra $A_{4,8}^0$ can be presented as follows in a way suitable for Lie reduction

$$\langle\mathcal{D} + a\mathcal{G}\rangle, \quad \langle\mathcal{G} + \delta\partial_t\rangle, \quad \langle\partial_t + \delta\partial_x\rangle, \quad \langle\partial_x\rangle, \tag{7}$$

where $a \in \mathbb{R}$ and $\delta \in \{0, 1\}$.

4 Exact solutions of the system of modified shallow-water equations

We now present the associated Lie reductions obtained using the optimal list of one-dimensional subalgebras of $\mathfrak{g}$ given in Theorem 1. We solve the arising systems of ordinary differential equations whenever possible.

(i) Subalgebra $\langle \mathcal{D} + a\mathcal{G} \rangle$. The reduction ansatz in this case is $u = \tilde{u}(p) + a \ln t$, $h = \tilde{h}(p)$, where $p = x/t - a \ln t + a$ is the new independent variable. Plugging this ansatz into system (1) reduces the two equations to the system of nonlinear ordinary differential equations

$$a - p\tilde{u}' + \tilde{u}\tilde{u}' + \left(1 + \frac{H}{\tilde{h}}\right)\tilde{h}' = 0, \quad -p\tilde{h}' + \tilde{u}\tilde{h}' + \tilde{h}\tilde{u}' = 0,$$

where here and in the following a prime denotes the derivative with respect to p. As this equation is still too complicated to be solved in closed form, one could resort to numerical integration for finding a solution of the above nonlinear system and then extending the numerical solution to a solution of the original modified shallow-water equations.

(ii) Subalgebra $\langle \mathcal{G} \rangle$. The ansatz for Lie reduction in this case is $u = \tilde{u}(p) + x/t$, $h = \tilde{h}(p)$, where $p = t$. The system of modified shallow-water equations (1) then reduce to

$$\tilde{u}' + \frac{\tilde{u}}{p} = 0, \quad \tilde{h}' + \frac{\tilde{h}}{p} = 0.$$

The solution of this system is $\tilde{u} = c_1/p$ and $\tilde{h} = c_2/t$, where $c_1, c_2 \in \mathbb{R}$, giving rise to the Galilean invariant solution

$$u = \frac{x + c_1}{t}, \quad h = \frac{c_2}{t} \tag{8}$$

of the modified shallow-water equations (1).

(iii) Subalgebra $\langle \mathcal{L}(f,g) \rangle$. For this algebra a suitable reduction ansatz is $u = \tilde{u}(p)$, $h = \tilde{h}(p)$, where $p = fx - gt$. Implicitly differentiating this ansatz with respect to t and x and solving the resulting algebraic system for the required derivatives u_t, u_x, h_t and h_x yields

$$u_t = -\frac{g\tilde{u}_p}{D}, \quad u_x = \frac{f\tilde{u}_p}{D}, \quad h_t = -\frac{g\tilde{h}_p}{D}, \quad h_x = \frac{f\tilde{h}_p}{D},$$

where the precise expression for D is not required. Plugging this ansatz into the modified shallow-water equations leads to

$$-g\tilde{u}_p + f\tilde{u}\tilde{u}_p + f\left(1 + \frac{H}{\tilde{h}}\right)\tilde{h}_p = 0, \quad -g\tilde{h}_p + f\tilde{u}\tilde{h}_p + f\tilde{h}\tilde{u}_p = 0,$$

where $f = f(\tilde{u}, \tilde{h})$ and $g = g(\tilde{u}, \tilde{h})$. The reduction with respect to the algebra $\langle \mathcal{L}(f,g) \rangle$ includes several physically relevant solution ansatzes. For example, in the case of $f = 1$ and $g = \text{const}$ (which obviously is a solution of the system (5) as required), the reduction ansatz coincides with that of a traveling wave solution. Similarly, for $f = \text{const}$ and $g = 0$ (or $f = 0$ and $g = \text{const}$) we obtain the stationary (or space independent) reduced system. In the present case, all these reduced equations admit only the constant solution $u = c_1$ and $h = c_2$ and thus are not of great interest. Less trivial but implicit solutions can be found by choosing f and g in a way such that they explicitly depend on u and h.

We now review succinctly the Lie reductions and results obtained in [30]. As noted in the previous section, no optimal list of one-dimensional subalgebras of their algebra $\mathfrak{g}^1$ was used to systematically carry out reductions. This deficiency gave rise to overly complicated reduction

ansatzes in [30]. As a consequence, these ansatzes could lead to solutions which in fact can be mapped to each other using a symmetry transformation, i.e. that are equivalent.

Three Lie reductions were presented in [30]. The first reduction (Case Ia) is carried out with respect to the algebra $\langle a_1 \mathcal{D} + a_2 \mathcal{G} + a_3 \partial_x + a_4 \partial_t \rangle$, where $a_1 \neq 0$. In view of our classification results and the optimal system (7), it is obvious that this case is equivalent to our case (i), i.e. the algebra $\langle \mathcal{D} + a\mathcal{G} \rangle$. Thus, the ansatz found in [30] should reduce to the ansatz presented above for case (i) when $a_1 = 1$ and $a_3 = a_4 = 0$. However, this is not the case. In fact, it can be directly checked that the ansatz in [30] for this case is incorrect. The correct (but overly complicated) ansatz for the algebra $\langle a_1 \mathcal{D} + a_2 \mathcal{G} + a_3 \partial_x + a_4 \partial_t \rangle$ would be

$$p = \frac{a_1^2 x + a_1 a_3 - a_2 a_4}{a_1^2(a_1 t + a_4)} - \frac{a_2}{a_1^2} \ln(a_1 t + a_4), \quad u = \tilde{u}(p) - \frac{a_2}{a_1} \ln(a_1 t + a_4), \quad h = \tilde{h}(p).$$

The second reduction (Case Ib) rests on the algebra $\langle a_2 \mathcal{G} + a_3 \partial_x + a_4 \partial_t \rangle$. It is equivalent to our case (ii), i.e. it is needless to assume $a_3 \neq 0$, $a_4 \neq 0$. However, it is crucial to point out that one cannot put $a_4 = 0$ using only the weaker equivalence relation imposed by the adjoint action associated with the subgroup G^1 of G on the subalgebra $\mathfrak{g}^1$. This is why it is instructive to compare the second case in the optimal list of subalgebras of the maximal Lie invariance algebra $\mathfrak{g}$ given in Theorem 1 with the second case in the optimal list of subalgebras of the subalgebra $\mathfrak{g}^1$ in (7) to each other. On the other hand, even with the adjoint action of G^1 on $\mathfrak{g}^1$, it would be possible to set $a_3 = 0$.

The last reduction (Case II) employs the algebra $\langle a_2 \mathcal{G} + a_3 \partial_x \rangle$ for reduction. The associated group-invariant solution is equivalent to our case (ii) i.e. it is again needless to assume $a_3 \neq 0$, and the exact solution found in [30] can be obtained from the Galilean invariant solution (8) through re-scaling of t and x and shifting of t.

To sum up, the three reductions carried out in [30] are either incorrect (Case Ia) or equivalent to a solution that could be obtained from a simpler reduction ansatz (Case Ib and Case II). From the more general point of view, if one would seek only reductions based on the subalgebra $\mathfrak{g}^1$, the optimal list (7) should be used to find the proper reduction ansatzes.

So far, we have not used the property that (1) is linearized to (5) by the hodograph transformation. We now investigate the possibility of finding exact solutions of (5) that can be used to obtain exact solutions of the initial system of modified shallow-water equations (1). To this end, we start by combining the two equations of the system (5) to a single equation for f by excluding g, which reads

$$2f_h + h f_{hh} - \left(1 + \frac{H}{h}\right) f_{uu} = 0. \tag{9}$$

A solution for this equation can be found from the separation ansatz $f = f_1(u) f_2(h)$, which yields the following form for f,

$$f = \frac{1}{\sqrt{h}} \left(c_1 \sin\left(\sqrt{\frac{c}{H}} u\right) + c_2 \cos\left(\sqrt{\frac{c}{H}} u\right) \right) \left(c_3 J_b\left(2\sqrt{c\frac{h}{H}}\right) + c_4 Y_b\left(2\sqrt{c\frac{h}{H}}\right) \right),$$

where $c \neq 0$, $c_1, \ldots, c_4 \in \mathbb{R}$, J_b and Y_b are the Bessel functions of the first and second kind, respectively, and $b = \sqrt{1 - 4c}$. As it is quite intricate to obtain the corresponding solution for g from the system (5) using the above solution for f, we restrict ourselves to finding particular solutions of system (5) following from simpler forms of f.

To give one example, let us set $c = 3/16$, $c_2 = c_4 = 0$. The above solution then reduces to

$$f = \frac{c_1}{h^{3/4}} \sin\left(\sqrt{\frac{c}{H}} u\right) \sin\left(\sqrt{4c\frac{h}{H}}\right),$$

and the solution for g from the system (5) becomes

$$g = \frac{c_1 H}{h^{5/4}} \left(\sqrt{\frac{h}{H}} \sin\left(\sqrt{c\frac{h}{H}} \right) \left(\frac{1}{\sqrt{3}} \cos\left(\sqrt{4c\frac{h}{H}} u \right) + \right.\right.$$
$$\left.\left. + \frac{u}{\sqrt{H}} \sin\left(\sqrt{4c\frac{h}{H}} \right) \right) + \frac{h}{H} \cos\left(\sqrt{c\frac{h}{H}} u \right) \cos\left(\sqrt{4c\frac{h}{H}} \right) \right) + c_2.$$

This solution cannot be solved for $f = t$ and $g = x$ in terms of u and h globally, but could be solved (e.g. numerically) pointwise and would then yield a solution manifold for the original modified shallow-water equations. This solution is obviously inequivalent to the group-invariant solutions found above.

Another, somewhat simpler solution of Eq. (9) is

$$f = c_1 \frac{u}{h} + c_2 \frac{1}{h} + c_3 u + c_4$$

which, when we substitute it back into (5) yields the following solution for g,

$$g = c_1 \left(\frac{u^2}{h} + \frac{H}{h} - \ln h \right) + c_2 \frac{u}{h} + c_3 \left(\frac{1}{2} u^2 - H \ln h - h \right) + c_5$$

where again $c_1, \ldots, c_5 \in \mathbb{R}$. In the special case when $c_1 = c_3 = c_4 = c_5 = 0$, this solution becomes the Galilean invariant solution (8) of the modified shallow-water equations found above. If all the constants $c_1, \ldots, c_5$ are non-zero, it is again impossible to solve the above solutions for $f = t$ and $g = x$ globally in terms of u and h.

In the same way one could proceed to find particular solutions of the system (5) and then extend them to solution of the original modified shallow-water equations. We will not pursue this idea here further though.

5 Conclusion

In the present paper we have reconsidered the problem of studying the Lie symmetries of a system of modified shallow-water equations (1) recently investigated in [30]. We have derived that the maximal Lie invariance algebra of this system is in fact infinite dimensional, containing the algebra found in [30] as a subalgebra. The infinite dimensional part of the maximal Lie invariance algebra (6) indicates that the system (1) can be linearized using a hodograph transformation, the result of which is the system (5). This is not unexpected, as any homogenous first-order system of partial differential equations with two dependent and two independent variables can be linearized by a hodograph transformation, provided that the coefficients in the system are only dependent on the unknown functions. As in [30] only a finite dimensional symmetry algebra is found, the indication, provided by the infinite dimensional symmetry algebra, of the existence of a linearizing transformation is missed.

When it comes to determining exact solutions of the system (1) we observe some common errors in finding exact solutions of partial differential equations. These errors are discussed in [?, 29] in the study [30]. The algebra $\mathfrak{g}^1$ is not maximal as a symmetry algebra of the system (1), moreover no optimal list of inequivalent one-dimensional subalgebras of the algebra $\mathfrak{g}^1$ was constructed. The algebra $\mathfrak{g}^1$ is only four-dimensional and for four-dimensional Lie algebras the problem of classifying all inequivalent subalgebras is completely solved. If one omits the construction of such an optimal list of inequivalent subalgebras, prior to carrying out the Lie reductions, then one might end up with unnecessarily complicated reduction ansatzes, which then lead to needlessly complicated exact solutions for the system of partial differential equations (or for the reduced systems). We observed this with the exact solutions found in [30]. In

another light, carrying out Lie reductions without reference to the optimal list of inequivalent subalgebras can lead to equivalent exact solutions, i.e. solutions that are related through a symmetry transformation.

Finally, as the linearizing hodograph transformation was missed in [30] the authors could not construct exact solutions of the linearized modified shallow-water equations (5), some of which can be extended to exact (and non-Lie) solutions of the modified shallow-water equations (1). This is another possible source for finding exact solutions of system (1).

Acknowledgements

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