Control Theory and PDE's

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1 Introduction

Differential equations are extremely useful tools in modelling all sorts of dynamical systems. As mathematicians, studying them for their own sake is an entirely acceptable, and even laudable venture.

Unfortunately, many people and institutions that supply actual grant money have different sensibilities, and want 'results' that are 'useful' and 'practical'. Heresy at its most profane, I agree. But it's the way of the world, and one must persevere. Onwards and upwards, as they say.

Most human endeavours involve some degree of trying to control the behaviour of some system or another. That is, a common theme in human thought is, "given that I know how this system will behave, how can I *change* it , to make the system act in a way that I would prefer?"
What follows is then the perpetual cycle of human history between the two stages of "What harm could it do?" and "How could we have known?"

The main goal of Control Theory is to answer this first question for dynamical systems modelled by differential equations (the latter two are left for watchdog organizations and historians, respectively).

These notes are divided into four sections. The first is a brief introduction to the control problem, and a statement of some of the central ideas and tools used in establishing controllability (whatever that may mean). The second develops some of the theory of finite linear systems, in particular arriving at Kalman's controllability condition, which completely characterizes the controllability properties of such systems. The third is a development of parallel controllability results for the heat equations, while the fourth provides investigates how the theory of viscosity solutions to Hamilton-Jacobi equations can be used to build optimal controls.

2 Section One: Control versus Optimal Control

The astute reader may have noticed that 'Optimal Control' is comprised of two words, and that the first is a modifier of the second. There is a reason for this; Control Theory and Optimal Control Theory ask two different, but related, questions. Let's investigate.

An exceptionally (perhaps lamentably) general version of the optimal control problem is as follows: given some sort of dynamical system, together with some way to control it, which may be modelled as

$$\Lambda y = Bu \tag{*}$$

Where Λ and B are operators, specifying the model of the system, and how the control acts on the system, respectively, y is the state of the system, and u is the control, from some admissible set of controls, U.

We are additionally given some 'cost functional' J(u), which maps to the reals, expressing the 'cost of running the control u'. The question Optimal Control theory asks is to find a control u, in the space of admissible controls, such that

$$J(u) = inf_{u \in U} J(u)$$

The slight peccadillo here is the following: how ought we choose the set of admissible controls? What would we want out of these controls in the first place? Obviously, it would do us no good if we found some control that minimized the cost functional, but this control didn't produce the desired results.

There are many types of control problems, depending on the context of the system under study, and the objectives of the controllers. Three types are generally considered, it is convenient to introduce some notation here:

Definition 1. *Let U be a function space and let y_1 be some given initial data in some Hilbert space V, let T>0, then we define the* ***set of reachable states*** *in time T*

$$R(T; y_1) = \{y(T) : \text{ y is a solution of } (*), \text{ with u} \in \text{U}\}$$

Here, U will be chosen based on the properties desired of our control. Control theory generally considers three types of controllability:

Definition 2. *We say system (*) is* ***exactly controllable*** *in time T if for every initial data y_1, R(T;y_1)=V.*

Definition 3. *System (*) is said to be* ***approximately controllable*** *in time T if for every initial data y_1, R(T;y_1) is dense in V.*

Definition 4. *System (*) is said to be* ***null controllable*** *in time T if, for every initial data y_1, we have $0 \in R(T; y_1)$*

Depending on the type of controllability required by the application, the Optimal Control setting will take as its set of admissible controls the control space that permits the type of controllability needed by the application. The question of control theory is then "for what types of controls does this system have the desired controllability properties?"

3 Section Two: Finite Linear Systems

In order to introduce some of the concerns of control theory in a more concrete setting, let's develop the theory of controllability for finite, linear systems, which turns out to have an exceptionally elegant answer. In this section, we consider the system:

$$\begin{aligned} &y' = Ay + Bu \quad , \text{t} \in (0, T) \\ &y_0 = y(0) \in \mathbb{R}^n \end{aligned} \tag{1}$$

We have A $\in$ M(n,n), B $\in$ M(n,m), m$\leq$n.

The controllability problem for this class of systems has a satisfying algebraic answer. Our program for this section will be as follows; first, we will show the equivalence of all three types of controllability for finite linear systems, then we'll prove Kalman's rank condition, which characterizes exactly when a system is controllable by the rank of a certain matrix.

First, let's establish the link between exact and null controllability:
Note that linearity of this system immediately gives us an equivalence between null controllability and exact controllability, since, for T>0, if y(T)=$y_T \neq 0$ then we can define: z=y-x, solving the system:

$$\begin{aligned} z' &= Az + Bu \\ z(0) &= y(0) - x(0) \end{aligned}$$

Where x is a solution to the system:

$$\begin{aligned} x' &= Ax \\ x(T) &= y_T \end{aligned}$$

Then y(T)=y_T iff z(T)=0. So due to the linear, finite nature of the system, we have that null controllability and exact controllability are equivalent. When we examine the heat equation, we'll see that this is not the case in an infinite dimensional context.

In a similarly quick way, we can brush concerns of approximate controllability under the rug by noting that the set of reachable states is affine (explicitly writing y(T) out as a function of y(0) and u with the variation of constants formula makes this clear), and for finite n, the only dense affine subspace of $\mathbb{R}^n$ is $\mathbb{R}^n$, so in the finite case, approximate controllability is equivalent to exact controllability.

Let's concern ourselves now with characterizing exactly which systems are controllable (in any of these three senses).
It is often convenient to examine the properties a system that is strongly related to our prime system, that is the **adjoint system** of (1). Letting A^* be the adjoint of A, then we consider the adjoint system, which runs backward in time:

$$\begin{aligned} -\phi' &= A^*\phi, \quad \mathrm{t} \in (0, \mathrm{T}) \\ \phi(T) &= \phi_T \end{aligned} \tag{2}$$

This system gives rise to the dual notion to controllability, known as **observability**.

Definition 5. *The adjoint system is said to be* ***observable*** *in time T>0 if ∃c>0 such that*

$$\int_0^T |B^*\phi|^2 \,\mathrm{d}t \geq c|\phi(0)|^2$$

For all $\phi_T \in \mathbb{R}^n$, with ϕ the corresponding solution to the adjoint system.

The above inequality is often referred to as the **observation inequality**. The concept of observability makes concrete the general notion that "the action of the controls is sufficient to determine the state of the system". In particular, the observation inequality guarantees that the solution of the adjoint equation at t=0 is completely determined by the $B^*\phi$ term, which is the quantity observed through the control.

Let us now relate null controllability (and hence, exact controllability) to the adjoint system.

Lemma 1. *Given T>0, a control $u \in L^2(0,T)$, and an initial data point y_0, we have that y_T=0 iff*

$$\int_0^T \langle u, B^*\phi\rangle \, \mathrm{d}t + \langle y_0, \phi(0)\rangle = 0$$

Proof. Let ϕ_T be arbitrary in $\mathbb{R}^n$, and let ϕ be the corresponding solution in the adjoint system. Then, note that:

$$\langle x', \phi\rangle = \langle Ay + Bu, \phi\rangle$$
$$\langle -\phi', y\rangle = \langle \phi, Ay\rangle$$

Summing these two gives:

$$\langle y', \phi\rangle + \langle \phi', y\rangle = \langle Bu, \phi\rangle$$
$$\Rightarrow d/dt\langle y, \phi\rangle = \langle u, B^*\phi\rangle$$
$$\Rightarrow \int_0^T \langle u, B^*\phi\rangle \, \mathrm{d}t = \langle y_T, \phi_T\rangle - \langle y_0, \phi_0\rangle$$

and certainly y_T=0 iff $\int_0^T \langle u, B^*\phi\rangle \, \mathrm{d}t = 0$ □

We're now almost ready to make explicit the duality between observability and controllability, but before this, we need a short lemma that follows from the above condition:

Lemma 2. *Suppose the functional $J : \mathbb{R}^n \to \mathbb{R}^n$, defined by $J(\phi_T)=\frac{1}{2}\int_0^T |B^*\phi|^2 \, \mathrm{d}t + \langle y_0, \phi(0)\rangle$ has a minimizer $\hat{\phi_T}$ and let $\hat{\phi}$ be the corresponding solution to the adjoint system with final data $\hat{\phi_T}$. Then $u = B^*\hat{\phi}$ is a control of system (1) that drives initial data y_0 to 0. (That is, the solution to system (1) with control u and $y(0) = y_0$ has $y(T) = 0$.)*

Proof. If $\hat{\phi_T}$ is a minimizer for J, then we have

$$\lim_{h\to 0} \frac{J(\hat{\phi_T} + h * \phi_T) - J(\hat{\phi_T})}{h} = 0, \quad \forall \phi_\mathrm{T} \in \mathbb{R}^n$$
$$\Leftrightarrow \int_0^T \langle B^*\hat{\phi}, B^*\phi\rangle \, \mathrm{d}t + \langle y_0, \phi(0)\rangle = 0, \quad \forall \phi_\mathrm{T} \in \mathbb{R}^n$$
$$\Rightarrow u = B^*\hat{\phi} \text{ is a control for (1)}$$

□

Interestingly, even though we claimed that questions of controllability precede questions of optimality, this gives a hint that, often we may be able to establish controllability properties simply by finding a minimizer of the appropriate functional. It does bear mentioning however, that this control is not necessarily unique.

In order to find minimizers of a functional, we often employ the **Direct Method of the Calculusof Variations** (DMCV), which we state here for completeness:

Theorem 1. *Let H be a reflexive Banach space, K a closed convex subset of H and $\phi : K \to \mathbb{R}$ a function such that:*
1. ϕ is convex
2. ϕ is lower semi-continuous
3. ϕ If K in unbounded, then ϕ is coercive
Then ϕ attains its minimum in K.

Now, we're in the position to establish the correspondence between observability and optimality:

Theorem 2. *System (1) is exactly controllable in time T iff the adjoint system is observable in time T.*

Proof. (Observability $\Rightarrow$ controllability)
By the above lemma, if the adjoint system is observable in time T, then it suffices to show that $\forall y_0 \in \mathbb{R}^n$, the functional J has a minimum.

By the DMCV, since J is continuous, it suffices to show that J is coercive, but the observation inequality gives us that

$$\int_0^T |B^*\phi|^2 \,\mathrm{d}t \geq c|\phi(0)|^2 \quad \forall \phi_T \in \mathbb{R}^n$$

So that, certainly,

$$J(\phi_T) > \frac{c}{2}|\phi_T|^2 - |\langle y_0, \phi(0)\rangle|$$

Which gives us coercivity.

(Controllability $\Rightarrow$ Observability)
Suppose that system (1) is exactly controllable in time T, but the adjoint system is not observable in time T.
Certainly, then there exists a sequence $(\phi_T^k)_{k\geq 1} \subset \mathbb{R}^n$ with $|\phi_T^k| = 1$, $\forall k \geq 1$ and

$$\lim_{k\to\infty} \int_0^T |B^*\phi^k|^2 \,\mathrm{d}t = 0$$

Moreover, a subsequence $(\phi_T^{k_m})_{m\geq 1}$ converges to ϕ_T and $|\phi_T| = 1$. Letting ϕ denote the solution to the adjoint system with final data ϕ_T. We have

$$\int_0^T |B^*\phi|^2 \,\mathrm{d}t = 0$$

But the observability inequality tells us that for some c>0

$$\int_0^T |B^*\phi|^2 \,\mathrm{d}t \geq c|\phi(0)|^2$$

And, noting that $\phi(t) = e^{A^*(t-T)}\phi(0)$, we have that this is equivalent to

$$\int_0^T |B^*\phi|^2 \,\mathrm{d}t \geq k|\phi_T|^2$$

for some k>0, and since $\int_0^T |B^*\phi|^2 = 0$, we have that $\phi_T = 0$, which is a contradiction to the fact that $|\phi_T| = 1$. ☐

We need one more lemma that relies crucially on the finite nature of the system in order to establish Kalman's rank condition. It is the following:

Lemma 3. *To show observability, it suffices to show that* $B^*\phi(t) = 0$, $\forall t \in [0,T] \Rightarrow \phi_T = 0$

Proof. Suppose the above condition holds, then define the seminorm $|\phi_T|_* = (\int_0^T |B^*\phi|^2 \,\mathrm{d}t)^{1/2}$. Since $|.|_*$ is a seminorm, it is a norm iff the assumed condition holds. Thus, it is a norm. But all norms on $\mathbb{R}^n$ are equivalent, so $\exists$ C>0 such that

$$C|\phi_T| \leq |\phi_T|_*$$
$$\Leftrightarrow C^2|\phi_T|^2 \leq \int_0^T |B^*\phi|^2 \,\mathrm{d}t$$

And using once more that $\phi(t) = e^{A^*(t-T)}\phi(0)$, we have that

$$K|\phi_0|^2 \leq \int_0^T |B^*\phi|^2 \,\mathrm{d}t$$

Hence the system is observable. □

Having established the equivalence of observability and we're now in a position to prove Kalman's rank condition:

Theorem 3. *System (1) is exactly controllable for some time T>0 iff*

$$\mathrm{rank}[B, AB, ..., A^{n-1}B] = n$$

Moreover, if the system is controllable for some time T>0, then it is controllable for all time.

Proof. (⇒) Suppose that $rank[B, AB, ..., A^{n-1}B]$ <n, then the rows are linearly dependent, so there exists some v $\in \mathbb{R}^n$ with v≠0 and

$$\mathrm{v}^T[B, AB, ..., A^{n-1}B] = [v^T B, v^T AB, ..., v^T A^{n-1}B] = 0$$

Which gives that $v^T A^k B = 0$, for k=$\{0, 1, ..., n-1\}$
By the Cayley-Hamilton theorem, we have the existence of a polynomial q, such that q(A)=0 and thus, there exist constants $c_1, ..., c_n$ such that

$$A^n = c_1 A^{n-1} + ... + c_n I$$

but by the above, multiplying by v^T gives $v^T A = 0$.
So that, in fact, we have $v^T A^k B = 0$, $\forall k \in \mathbb{N}$.
This gives us that $v^T e^{At} B = 0$, $\forall t$,
but the variation of constants formula tells us that the state of the system at any point is given by:

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s$$

Taking the inner product against v kills the integrand, giving:

$$\langle v, y(T)\rangle = \langle v, e^{aT}y_0\rangle$$

So the projection of the solution along v is independent of the control, hence the system is uncontrollable.

(⇐) Suppose now that $rank([B, AB, ..., A^{n-1}B]) = n$, we know that it suffices to show that the system is observable. For which it suffices to show that $B^*\phi(t) = 0$, $\forall t \in [0, T] \Rightarrow \phi_T = 0$.

So, suppose $B^*\phi = 0$.
Since $\phi(t) = e^{A^*(T-t)}\phi_T$, we have that $0 = B^* e^{A(T-t)}\phi_T$, for all $0 \leq t \leq T$.
Taking derivatives in T yields that $B^*[A^*]\phi_T = 0$ for all k>0.
This implies that $[B^*, B^*A^*, ..., B^*(A^*)^{n-1}]\phi_T$=0, but since $[B, AB, ..., A^{n-1}B]$ is of full rank, so is $[B^*, B^*A^*, ..., B^*(A^*)^{n-1}]$, hence ϕ_T=0. Which is our desired result. □

4 Section Three: Controllability of the Heat Equation

As an example of how one extends the techniques and concerns of the finite linear case to the infinite case, where exact, approximate and null controllability are no longer equivalent, let us consider now the heat equation: Let $\Omega \subset \mathbb{R}^n$ and $\Gamma = \partial\Omega$, T>0 we consider:

$$\begin{aligned} &y_t - \triangle y = u \ \text{ in } \Omega \times (0,T) \\ &y = 0 \ \ on\ \Gamma \times (0,T) \\ &y(x,0) = y_0(x) \ \ in\ \Omega \end{aligned}$$

With $\text{supp}(u) := \omega \subset \Omega$. This is known (for obvious reasons) as the *interior control problem.* We consider here the questions of approximate and exact controllability, as the question of null controllability, while answered in the affirmative, is considerably more involved.

Let us consider the problem of exact controllability (which is resolved exceptionally quickly):

Theorem 4. *The heat equation is not exactly controllable for any $\omega \subsetneq \Omega$, $T > 0$.*

Proof. Note that for any $T > 0$, solutions of the heat equation are smooth on $\Omega \setminus \omega$, so for any target $v(x) \in L^2(\Omega) \setminus C^\infty(\Omega)$ we have that $y(x,T) \neq v(x)$, so that the system is not exactly controllable. □

Note: In the case that $\omega = \Omega$, we have by standard existence and uniqueness of the PDE that the system is exactly controllable in $H^1(\Omega)$.

Lemma 4. *The heat equation is approximately controllable for any T>0*

We give two proofs of this fact, the first is quick but unconstructive, but proves approximate controllability for a more general class of systems, while the second involves the construction of an explicit control for the heat equation, and is somewhat more enlightening. We will recall here a consequence of Holmgren's Uniqueness theorem, that if P is an elliptic partial differentiation operator with analytic coefficients, and Py is real analytic in some open neighbourhood of $\Omega \subset \mathbb{R}^n$, then y is analytic.

For the first proof, recall that the Hahn-Banach theorem implies that if A is a subspace of X, and 0 is the only element in the orthogonal complement of A, then A is dense in X.

Proof. Consider a system of the form

$$\begin{aligned} \partial_t y + Ay = f + u \quad &\text{in } \Omega \times (0,T) \\ y = 0 \quad &\text{on } \Gamma \times (0,T), \\ y(x,0;u) = y_0(x) \quad &\text{for } x \in \Omega \end{aligned}$$

With

$$\begin{aligned} &a_{ij} \in L^\infty(\Omega \times (0,T)) \\ &\sum_{i,j} a_{ij}\xi_i\xi_j \geq \alpha\|\xi\|^2, \quad \alpha > 0, \xi \in \mathbb{R}^n \end{aligned}$$

Note first that by a translation, it suffices to consider $f = 0$, $y_0 = 0$
Now, let $\psi \in L^2([0,T] \times \Omega)$ be orthogonal to the subspace generated by y(v) as v runs over $L^2([0,T]\times\Omega)$, with y the solution to the above system, with control v, and fixed but arbitrary initial data. We have:

$$\int_{[0,T]\times\Omega} y(v)\psi \, \mathrm{d}x\mathrm{d}t, \quad \forall v \in L^2([0,T] \times \Omega)$$

We then let ξ be the solution to the following adjoint system:

$$-\frac{d\xi}{dt} + A^*\xi = \psi \quad \text{in } [0,T] \times \Omega$$
$$\xi = 0 \quad on\ [0,T] \times \Sigma$$
$$\xi(x,T) = 0 \quad \text{on } \Omega$$

Then we have:

$$\int_{[0,T]\times\Omega} y(v)\psi \,\mathrm{d}x\mathrm{d}t = \int_{[0,T]\times\Omega} y(v)(-\frac{d\xi}{dt} + A^*\xi)\,\mathrm{d}x\mathrm{d}t$$
$$\int_{[0,T]\times\Omega} (\frac{dy(v)}{dt} + Ay(v))\xi \,\mathrm{d}x\mathrm{d}t$$
$$\int_{[0,T]\times\Omega} \xi v \,\mathrm{d}x\mathrm{d}t, \quad \forall v \in L^2([0,T]\times\Omega)$$

But then $\xi = 0$ in $[0,T]\times\Omega$ implying that $\psi = 0$, so that by the Hahn-Banach theorem, the reachable set is dense in $L^2([0,T]\times\Omega)$, and so the system is approximately controllable. □

Note that the above proof holds for a much more general class of functions that just the heat equation, so we've actually proved something much stronger. This proof is, however, somewhat unsatisfying, as it gives so intuition as to how to construct the desired control. To that end, let's examine the following constructive proof of the approximate controllability of the heat equation. We first define the functional J_ϵ:$L^2(\Omega) \longrightarrow \mathbb{R}$

$$J_\epsilon(\phi_T) = \frac{1}{2}\int_Q \phi^2 \,\mathrm{d}x\mathrm{d}t + \epsilon\|\phi_T\|_{L^2(\Omega)} - \int_\Omega y_1\phi_T \,\mathrm{d}x$$

Where ϕ is the solution to the adjoint equation with final data ϕ_T, and $Q = \omega \times (0,T)$, and $y_1 \in L^2(\Omega)$ is the final target state. Suppose $\epsilon > 0$ is given, and by the linearity of the system, we may suppose that $y_0 = 0$, as in the last proof.
The idea is that the minimizer of this functional, should it lie in $L^2(\Omega)$ will be our desired control.

Lemma 5. *If $\hat{\phi_T}$ is a minimum of J_ϵ in $L^2(\Omega)$ and $\hat{\phi}$ is the solution of the adjoint system with ϕ_T as final data, then $u = \hat{\phi}|_\omega$ is an approximate control for the heat equation.*
(ie: $\|y(T) - y_1\|_{L^2(\Omega)} \leq \epsilon$)

Proof. Suppose that $\hat{\phi_T} \in L^2(\Omega)$ is a minimum of the functional J_ϵ. Then, for any $\psi_0 \in L^2(\Omega)$ and h $\in \mathbb{R}$, we have
$J_\epsilon(\hat{\phi}) \leq J_\epsilon(\hat{\phi} + h\psi_0)$
Writing $J_\epsilon(\hat{\phi} + h\psi_0)$ out explicitly, we have:

$$J_\epsilon(\hat{\phi} + h\psi_0) = \frac{1}{2}\int_Q |\hat{\phi} + h\psi_0|^2 \,\mathrm{d}x\mathrm{d}t + \epsilon\|\hat{\phi} + h\psi_0\|_{L^2(\Omega)} - \int_\Omega y_1\hat{\phi} + h\psi_0 \,\mathrm{d}x$$
$$= \frac{1}{2}\int_Q \hat{\phi}^2 \,\mathrm{d}x\mathrm{d}t + \frac{h^2}{2}\int_Q \psi^2 \,\mathrm{d}x\mathrm{d}t + h\int_Q \hat{\phi}\psi \,\mathrm{d}x\mathrm{d}t + \epsilon\|\hat{\phi_T} + h\psi_0\|_{L^2(\Omega)} - \int_\Omega y_1(\hat{\phi_T} + h\psi_0)\,\mathrm{d}x$$

So that $0 \leq J_\epsilon(\hat{\phi} + h\psi_0) - J_\epsilon(\hat{\phi})$ gives us:

$$
\begin{aligned}
0 \leq & \int_Q \hat{\phi}^2 \,\mathrm{d}x\mathrm{d}t + \frac{h^2}{2}\int_Q \psi^2 \,\mathrm{d}x\mathrm{d}t + h\int_Q \hat{\phi}\psi \,\mathrm{d}x\mathrm{d}t + \epsilon\|\hat{\phi}_T + h\psi_0\|_{L^2(\Omega)} - \int_\Omega y_1(\hat{\phi}_T + h(\psi_0)\,\mathrm{d}x + \int_\Omega y_1\hat{\phi}_T \,\mathrm{d}dx \\
& - \frac{1}{2}\int_Q \hat{\phi}^2 \,\mathrm{d}x\mathrm{d}t + \epsilon\|\hat{\phi}_T\|_{L^2(\Omega)} \\
= & \epsilon(\|\hat{\phi}_T + h\psi_0\|_{L^2(\Omega)} - \|\hat{\phi}_T\|_{L^2(\Omega)}) + \frac{h^2}{2}\int_Q \psi^2 \,\mathrm{d}x\mathrm{d}t + h(\int_Q \hat{\phi}\psi \,\mathrm{d}x\mathrm{d}t - \int_\Omega y_1\psi_0 \,\mathrm{d}x)
\end{aligned}
$$

The triangle inequality on the L^2-norm gives us that

$$
0 \leq \epsilon|h|\|\psi_0\|_{L^2(\Omega)} + \frac{h^2}{2}\int_Q \psi^2 \,\mathrm{d}x\mathrm{d}t + h(\int_Q \hat{\phi}\psi \,\mathrm{d}x\mathrm{d}t - \int_\Omega y_1\psi_0 \,\mathrm{d}x)
$$

Dividing through by h>0 and taking the limit, and then doing similarly for h<0 in the above inequality gives us that

$$
|\int_Q \hat{\phi}\psi \,\mathrm{d}x\mathrm{d}t - \int_\Omega y_1\psi_0 \,\mathrm{d}x| \leq \epsilon\|\psi_0\|_{L^2(\Omega)}, \quad \forall\psi_0 \in L^2(\Omega)
$$

We now want to relate the first term in the left hand side to our approximate solution. To do so, remark that

$$
y_t - \triangle y = u|_\omega
$$

So, setting $u = \hat{\phi}$, and multiplying through by ψ, the solution to the adjoint system with ψ_0 as final data, we have

$$
\begin{aligned}
\int_\omega \hat{\phi}\psi \,\mathrm{d}x\mathrm{d}t &= \int_Q (y_t - \triangle y)\psi \,\mathrm{d}x\mathrm{d}t \\
&= \int_\Omega y\psi|_0^T - \int_0^T\int_\Omega \triangle u\psi \\
&= \int_\Omega y\psi|_0^T - \int_0^T\int_\Omega y\triangle\psi \\
&= \int_\Omega (y\psi)|_0^T - \int_\Omega\int_0^T y(\psi_T + \triangle\psi) \\
&= \int_\Omega (y(T) - y_0)\psi_T \\
&= \int_\Omega y(T)\psi_T
\end{aligned}
$$

So that

$$
|\int_\Omega (y(T) - y_1)\psi_0 \,\mathrm{d}x| \leq \epsilon\|\psi_0\|_{L^2(\Omega)}, \quad \forall\psi_0 \in L^2(\Omega)
$$

Taking $\psi_0 = y(T) - y_1$ guarantees us that

$$
\|y(T) - y_1\|_{L^2(\Omega)} \leq \epsilon
$$

Which proves our result. $\square$

Now, of course, we've assumed the existence of a minimizer of this functional in L^2. Once we've confirmed this, we've established approximate controllability with the control constructed above.

Lemma 6. *$\exists \hat{\phi}_T \in L^2(\Omega)$ such that $J_\epsilon(\hat{\phi}_T) = min_{\phi_t \in L^2(\Omega)} J_\epsilon(\phi_T)$*

Proof. We want to apply the Direct Method in the Calculus of Variations. Notice that the bound we established in the previous proof on $J_\epsilon(\hat{\phi}_T + h\psi) - J_\epsilon(\hat{\phi}_T)$ gives us continuity of the operator, and convexity follows immediately from the linearity of integration and the triangle inequality from the L^2-norm. So we just need to establish coercivity of the operator. To do this, it suffices to show

$$\liminf_{\|\phi_T\|_{L^2(\Omega)} \to \infty} \frac{J_\epsilon(\phi_T)}{\|\phi_T\|_{L^2(\Omega)}} \geq \epsilon$$

We take an arbitrary sequence $\phi_{T,j} \subset L^2(\Omega)$ and normalize them such that

$$\tilde{\phi}_{T,j} = \frac{\phi_{T,j}}{\|\phi_{T,j}\|_{L^2(\Omega)}}$$

and let $\tilde{\phi}_j$ be the solution to the adjoint system with final data $\tilde{\phi}_{T,j}$. Then,

$$\frac{J_\epsilon(\phi_{T,j})}{\|\phi_{T,j}\|_{L^2(\Omega)}} = \frac{1}{2}\|\phi_{T,j}\|_{L^2(\Omega)} \int_{[0 \times T] \times \omega} \tilde{\phi}_j^{\,2} \, \mathrm{d}x\mathrm{d}t + \epsilon - \int_\Omega y_1 \tilde{\phi}_{T,j} \, \mathrm{d}x$$

There are two possible cases now:
1. $\liminf_{j\to\infty} \int_{[0,T]\times\omega} |\tilde{\phi}_j|^2 \, \mathrm{d}x\mathrm{d}t > 0$, in which case, since the $\int_\Omega y_1 \tilde{\phi}_{T,j} \, \mathrm{d}x$ term is bounded by the L^2 norm of y_1 due to Holder, we have $\liminf_{j\to\infty} J_\epsilon(\phi_{T,j}) = \infty$, so we have coercivity directly.

2. $\liminf_{j\to\infty} \int_{[0,T]\times\omega} |\tilde{\phi}_j|^2 \, \mathrm{d}x\mathrm{d}t = 0$
In this case we note that since $\tilde{\phi}_{T,j}$ is bounded in $L^2(\Omega)$ so there exists a weakly convergent subsequence $\tilde{\phi}_{T,j} \to \psi_0$ such that $\tilde{\phi}_j \to \psi$ weakly, where ψ is the solution of the adjoint system with final data ψ_0.
By lower semi-continuity we have:

$$\int_{[0,T]\times\omega} \psi^2 \, \mathrm{d}x\mathrm{d}t \leq \liminf_{j\to\infty} \int_{[0,T]\times\omega} \tilde{\phi}_j^{\,2} \, \mathrm{d}x\mathrm{d}t$$

So that $\psi = 0$ on $[0,T] \times \omega$, and Holmgren's uniqueness theorem guarantees that $\psi = 0$ on $[0,T] \times \Omega$, so $\psi = 0$.
Therefore, $\tilde{\phi}_{T,j} \to 0$ weakly in $L^2(\Omega)$ so that $\int_\Omega y_1 \tilde{\phi}_{T,j} \, \mathrm{d}x \to 0$ as well, thus we have that $\liminf_{j\to\infty} \frac{J_\epsilon(\phi_{T,j})}{\|\phi_{T,j}\|_{L^2(\Omega)}} \geq \liminf_{j\to\infty}(\epsilon - \int_\Omega y_1 \tilde{\phi}_{T,j} \, \mathrm{d}x) = \epsilon$
as desired.

□

5 Section Four: Optimal Control and Hamilton-Jacobi Equations

We would like to approach the problem of constructing optimal controls. To that end, consider the following problem:

$$\dot{x}(s) = f(x(s), \alpha(s)) \;\; (t < s < T)$$
$$x(t) = x \in \mathbb{R}^n$$

With α a control, and we take the space of admissible controls, U, to be the set of measurable functions on [0,T], with T>0 a fixed time. We define the **cost functional** to be:

$$C_{x,t}(\alpha) = \int_t^T L(x(s), \alpha(s))\,\mathrm{d}s + \psi(x(T))$$

And our goal is to find some α that minimizes this functional. In our current setting, we suppose that f, L and ψ are all bounded and Lipschitz continuous on their domain of definition. As it turns out, there is a useful result that gives us a starting place for this endeavour, giving necessary conditions for such a control to be optimal. It is the Pontryagin Maximum Principle, which we state here:

Theorem 5. *Given the control system*

$$\dot{y} = f(y,u), \;\; u(t) \in U, \;\; t \in [0,T], \;\; y(0) = y_0, \;\; \phi_i(y(T)) = 0, \;\; i = 1, ..., m, \;\; m \leq n$$

Consider the problem of finding a control u^* such that

$$\psi(y(T,u^*)) = \max_{u \in U} \psi(y(T,u))$$
$$\psi : \mathbb{R}^n \to \mathbb{R}$$

Define y^* to be the trajectory of the state under the optimal control. Then, there exists a non-zero vector function p(t) such that:

$$p(T) = \lambda_0 \nabla\psi(y^*(T)) + \sum_{i=1}^{m} \lambda_i \nabla\phi_i(y^*(T)) \;\; with \;\; \lambda_0 \geq 0$$
$$\dot{p}(t) = -p(t) D_y f(t, y^*(t), u^*(t)), \;\; t \in [0,T]$$
$$p(\tau) f(\tau, y^*(\tau), u^*(\tau)) = max_{w \in U}\{p(\tau) f(\tau, y^*(\tau), w)\} \;\; fora.e.\tau \in [0,T]$$

For some λ_i.

Note that we can actually use this theorem in the more general optimization problem:

$$\dot{y} = f(y,u), \;\; u(t) \in U, \;\; t \in [0,T], \;\; y(0) = y_0$$

With control u maximizing:

$$\max_{u \in U}\{\psi(y(T,u)) - \int_0^T L(t, y(t), u(t))\,\mathrm{d}t\}$$

by introducing y_{n+1} such that:

$$\dot{y}_{n+1} = L(t, y(t), u(t))$$
$$y_{n+1}(0) = 0$$

And considering the re-defined problem:

$$\max_{u \in U}\{\psi(y(T,u)) - y_{n+1}(T,u)\}$$

Obviously, we can equally well consider minimization problems with this technique.
The Pontryagin Maximum Principle provides us with necessary conditions for a control to be optimal, and so it helps us search for optimal controls. But how ought we check the optimality of the controls once we've determined a candidate? This is where Hamilton-Jacobi-Bellman equation come into play:

Let give the context in which we will be working.

Definition 6. *Given a Hamilton-Jacobi-Bellman equation of the form*

$$u_t + min_{a \in A}\{f(x,a) * Du + h(x,a)\}) = 0 \ \ in\ \mathbb{R}^n \times [0,T]$$
$$u = g \quad on\ \mathbb{R}^n \times \{t = T\}$$

A function u is called a **viscosity solution** *to the above problem if $u = g$ on $\mathbb{R}^n \times \{t = T\}$, and for all $v \in C^\infty(\mathbb{R}^n \times (0,T))$*

$$\text{if u} - \text{v has a local maximum at } (x_0, t_0) \in \mathbb{R}^n \times (0,T) \text{ then}$$
$$v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \geq 0$$

And

$$\text{if u} - \text{v has a local minimum at } (x_0, t_0) \in \mathbb{R}^n \times (0,T) \text{ then}$$
$$v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \leq 0$$

The utility of viscosity solutions comes from the following fact:

Theorem 6. *If a Jacobi-Hamilton PDE satisfies the following Lipschitz condition:*

$$|H(p,x) - H(q,x)| \leq C|p - q|$$
$$|H(p,x) - H(p,y)| \leq C|x - y|(1 + |p|)$$

for x,y,p,q $\in \mathbb{R}^n$, then the initial value problem of the PDE has at most one viscosity solution.

Our strategy for obtaining sufficiency conditions for controls generated through the Pontryagin Maximum Principle will be the following:
first, we define the **value function**:

$$u(x,t) = \inf_{\alpha \in U}\{\int_t^T L(x(s), \alpha(s))\,\mathrm{d}s + \psi(x(T))\}$$

Our goal will be to show that this function is, in fact, the unique viscosity solution to the Hamilton-Jacobi-Bellman equation, thereby giving us a condition against which we can check potential controls.

Our first task is to establish a useful lemma, which states that the optimal control must be optimal on subinterval, which allows us to break up it's computation. This is known as the **Dynamic Programming Principle**:

Lemma 7. *For all $\tau \in [t,T]$ we have:*

$$u(x,t) = \inf_{\alpha \in U}\{\int_t^\tau L(x(s), \alpha(s))\,\mathrm{d}s + u(x(\tau), \tau)\}$$

Proof. We split the problem up. First, choose some arbitrary control $\alpha_1 \in U$ and solve

$$\dot{x_1}(s) = f(x_1(s), \alpha_1(s)) \ \ (t < s < \tau)$$
$$x_1(t) = x$$

Fix $\epsilon > 0$ and choose $\alpha_2 \in U$ such that

$$\int_\tau^T L(x_2(s), \alpha_2(s))\,\mathrm{d}s + \psi(x_2(T)) \leq u(x_1(\tau), \tau)$$

where

$$\dot{x_2}(s) = f(x_2(s), \alpha_2(s)) \;\; (t < s < \tau)$$
$$x_2(t) = x$$

Then we set $\alpha_3(s) = \alpha_1(s)$ on (t, τ) and $= \alpha_2$ on (τ, T)
The fact that $u(x,t) = \inf_{\alpha \in U} C_{x,t}(\alpha)$ gives us that

$$\begin{aligned}u(x,t) &\leq C_{x,t}(\alpha_3)\\ &\int_t^\tau L(x_1(s), \alpha_1(s))\, \mathrm{d}s + \int_\tau^T L(x_2(s), \alpha(s))\, \mathrm{d}s + \psi(x_2(T))\\ &\geq \int_t^\tau L(x_1(s), \alpha_1(s))\, \mathrm{d}s + u(x_1(\tau), \tau) + \epsilon, \;\; \text{by our choice of } \alpha_2\end{aligned}$$

But α_1 was arbitrary, so

$$(u(x,t)) \leq \inf_{\alpha \in U} \{ \int_t^\tau L(x(s), \alpha(s))\, \mathrm{d}s + u(x(\tau), \tau) \} + \epsilon$$

In the other direction, fix again $\epsilon > 0$ and let $\alpha_4 \in U$ be such that

$$u(x,t) + \epsilon \geq \int_t^T L(x_4(s), \alpha_4(s))\, \mathrm{d}s + \psi(x_4(T))$$

Where,

$$\dot{x_4}(s) = f(x_4(s), \alpha_4(s)) \;\; (t < s < \tau)$$
$$x_4(t) = x$$

But certainly,

$$\begin{aligned}u(x_4(\tau)) &\leq \int_\tau^T L(x_4(s), \alpha_4(s))\, \mathrm{d}s + \psi(x_4(T)), \;\; \text{So taking the infimum of both sides}\\ u(x,t) + \epsilon &\geq \inf_{\alpha \in U} \{ \int_t^\tau L(x(s), \alpha(s))\, \mathrm{d}s + u(x(\tau), \tau) \}\end{aligned}$$

Taking $\epsilon \to 0$ in both inequalities finishes the proof. □

We now need to establish that the value function is Lipschitz continuous:

Lemma 8. *Suppose that f, L, and ψ are bounded and Lipschitz continuous, then the value function u(x,t) is bounded and Lipschitz continuous. That is, there exists a constant K, such that*

$$|u(x,t)| \leq K$$
$$|u(x,t) - u(x', t')| \geq K(|x - x'| + |t - t'|)$$

Proof. Boundedness of u is clear from the boundedness of the component functions. So we focus on Lipschitz continuity.
Let $(\bar{x}, \bar{t})$ be given initial data, and let $\epsilon > 0$ be given. Choose a control $w \in U$ such that

$$C_{\bar{x}, \bar{t}}(w) \leq u(\bar{x}, \bar{t}) + \epsilon$$

let y(t) be the trajectory under w. Define then z(t) to be the trajectory under different initial data $(\hat{x}, \hat{t})$ but keeping the same control. Since f is bounded and Lipschitz, we have:

$$
\begin{aligned}
&|y(\hat{t}) - z(\hat{t})| \leq |y(\hat{t}) - y(\bar{t})| + |y(\bar{t}) - z(\hat{t})| \leq C|\hat{t} - \bar{t}| + |\bar{x} - \hat{x}|, \;\; \textit{and} \\
&|y(t) - z(t)| \leq e^{C|t-\hat{t}|}|y(\hat{t}) - z(\hat{t})|, \;\; \textit{by Gronwall's inequality} \\
&\leq e^{CT}(C|\bar{t} - \hat{t}| + |\bar{x} - \hat{x}|)
\end{aligned}
$$

Using next the boundedness and Lipschitz continuity of L and ψ, we have

$$
\begin{aligned}
&C_{\hat{x},\hat{t}}(w) = C_{\bar{x},\bar{t}}(w) + \int_{\hat{t}}^{\bar{t}} L(z,w)\,\mathrm{d}t + \int_{\bar{t}}^{T} L(z,w) - L(y,w)\,\mathrm{d}t + \psi(z(T)) - \psi(y(T)) \\
&\leq C_{\bar{x},\bar{t}}(w) + K|\hat{t} - \bar{t}| + \int_{\bar{t}}^{T} K|z(t) - y(t)|\,\mathrm{d}t + K|z(T) - y(T)| \\
&\leq C_{\bar{x},\bar{t}}(w) + C'|\hat{t} - \bar{t}| + |\bar{x} - \hat{x}|, \;\; \text{so} \\
&u(\hat{x},\hat{t}) \leq \;C_{\hat{x},\hat{t}}(w) \leq u(\bar{x},\bar{t}) + \epsilon + C'(|\hat{t} - \bar{t}| + |\bar{x} - \hat{x}|)
\end{aligned}
$$

We then let $\epsilon \to 0$ and repeat the argument with the roles of $(\hat{x}, \hat{t})$ and $(\bar{x}, \bar{t})$ reversed, which yields Lipschitz continuity of u. □

We are now ready to state the main result of this section, that is:

Theorem 7. *The value function u is the unique viscosity solution of the terminal value problem for the Hamilton-Jacobi-Bellman equation:*

$$
\begin{aligned}
&u_t + \min_{\alpha \in U}\{f(x,\alpha)Du + L(x,\alpha)\} \\
&u = \psi \quad \textit{on } \mathbb{R}^n \times \{t = T\}
\end{aligned}
$$

Proof. Obviously, from our bounds on f, and ψ, H satisfies the necessary Lipschitz conditions, so that it has at most one viscosity solution. We aim to show that u(x,t) is this very solution.

Certainly, by construction, $u(x,T) = \psi(x)$, so letting $\phi \in C^1((R)^n \times (0,T))$, we have two things to show:

If $u - \phi$ attains a local maximum at (x_0, t_0), then
$\phi_t(x_0,t_0) + \min_{w \in U}\{f(x_0,w)D\phi(x_0,t_0) + L(x_0,w)\} \geq 0$

And

If $u - \phi$ attains a local minimum at (x_0, t_0), then
$\phi_t(x_0,t_0) + \min_{w \in U}\{f(x_0,w)D\phi(x_0,t_0) + L(x_0,w)\} \leq 0$

We begin with the first.
By translation and restricting our attention to a neighbourhood of (x_0, t_0) we can may assume $u(x_0,t_0) = \phi(x_0,t_0)$ and $u(x,t) \leq \phi(x,t)$ otherwise. Now, we suppose that there exists $w \in U$ and $\theta > 0$ such that

$$
\phi_t(x_0,t_0) + D\phi(x_0,t_0)f(x_0,w) + L(x_0,w) < -\theta \tag{1}
$$

That is, the first condition does not hold. But if this is the case, then by continuity and Lipschitz continuity, there is some δ such that while $|t - t_0| \leq \delta|$ and $|x - x_0| \leq C\delta$, we have

$$
\phi_t(x,t) + D\phi(x,t)f(x,w) < -\theta - L(x,w)
$$

Let y(t) be the trajectory under w, that is, the solution solution of:

$$\dot{y}(t) = f(t, x)$$
$$y(t_0) = x_0$$

Then we have

$$\begin{aligned}
u(t_0 + \delta, y(t_0 + \delta)) - u(t_0, x_0) &\leq \phi(y(t_0 + \delta), t_0 + \delta) - \phi(x_0, t_0) \\
&= \int_{t_0}^{t_0+\delta} \frac{d}{dt}\phi(y(t, y))\, \mathrm{d}t \\
&= \int_{t_0}^{t_0+\delta} \phi_t(x(t), t) + D\phi(x(t), t) f(x(t), w)\, \mathrm{d}t \\
&\leq - \int_{t_0}^{t_0+\delta} L(x(t), w)\, \mathrm{d}t - \delta\theta \quad \text{So that} \\
u(x_0, t_0) &\geq \int_{t_0}^{t_0+\delta} L(x(t), w)\, \mathrm{d}t + u(t_0 + \delta, y(t_0 + \delta)) + \delta\theta
\end{aligned}$$

But $u(x_0, t_0) = \int_{t_0}^{t_0+\delta} L(x(t), w\, \mathrm{d}t + u(x(t_0 + \delta), t_0 + \delta)$ by the Dynamic Programming Principle, so we have a contradiction.

Now we prove the second condition for u to be a viscosity solution. Again, by translation and restriction of attention, we may suppose WLOG that

$$u(x_0, t_0) = \phi(x_0, t_0) \;\text{ and}$$
$$u(x, t) \geq \phi(x, t) \;\; \forall x, t$$

Suppose that the second condition fails, that is, that $\exists \theta > 0$ such that

$$\phi(x_0, t_0) + D\phi(x_0, t_0) f(x_0, w) + L(x_0, w) > \theta, \; \forall w \in U$$

By continuity again, we have for x and t such that $|t - t_0| \leq \delta$ and $|x - x_0| \leq C\delta$, for some $\delta > 0$,

$$\phi(x_0, t_0) + D\phi(x_0, t_0) f(x_0, w) > \theta - L(x_0, w)\; \forall x \in U$$

Choosing an arbitrary control $\beta \in U$ and letting x(t) denote the corresponding trajectory of the system, we have:

$$\begin{aligned}
u(t_0 + \delta, y(t_0 + \delta)) - u(t_0, x_0) &\geq \phi(y(t_0 + \delta), t_0 + \delta) - \phi(x_0, t_0) \\
&= \int_{t_0}^{t_0+\delta} \frac{d}{dt}\phi(y(t, y))\, \mathrm{d}t \\
&= \int_{t_0}^{t_0+\delta} \phi_t(x(t), t) + D\phi(x(t), t) f(x(t), w)\, \mathrm{d}t \\
&\geq \int_{t_0}^{t_0+\delta} \theta - L(x(t), w)\, \mathrm{d}t \quad \text{So that} \\
u(x(t_0 + \delta), t_0 + \delta_0) + \int_{t_0}^{t_0+\delta} L(x(t), u(t))\, \mathrm{d}t &\geq u(x_0, t_0) \;\; \forall w \in U
\end{aligned}$$

Taking the infimum over all controls and applying the Dynamic Programming Principle to the left hand side gives

$$u(x_0, t_0) \geq u(x_0, t_0) + \delta\theta$$

A contradiction, hence u the unique viscosity solution to the HJB equation. □

So we now have a sufficient condition against which we may check the optimality of controls, as promised.

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