Geometric intuition: from Hölder spaces to the Calderón-Zygmund estimate

A survey of Lihe Wang's paper

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Contents

Abstract

The following report is based on paper by Wang [**?**]. We include Wang's discussion of Hölder norms and supplement it with results from class and assignments. Then, we introduce the Hardy-Littlewood maximal function, the Calderòn-Zygmund decomposition, and the Vitali covering lemma. Finally, we include the proof of the Calderòn-Zygmund estimate

$$\int_{B_1} |D^2 u|^p \leq c \left(\int_{B_1} |f|^p + \int_{B_1} |u|^p \right).$$

1 Hölder spaces

1.1 Control on functions

Let us first discuss the geometry of Hölder spaces. We are motivated by the discussion in Wang. Hölder norms given us intuition on the density of a function and what can be said about sets

$$\{|f| > \lambda\} = \{x : |f(x)| > \lambda\}$$

and their measure. Recall that the definition of Hölder norm $C^{0,\alpha}$ for $\alpha \in (0, 1]$ is

$$\|u\|_{C^{0,\alpha}} = \sup_x |u(x)| + \sup_{x,y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}. \tag{1}$$

Note that, from homework assignment 4, question 5. (a), the space for which (**??**) holds for $\alpha = 1 + \varepsilon$, for any $\varepsilon > 0$, is only the space of constants. Moreover, from (b), this space is much larger than C^1; continuously differentiable functions are not dense in $C^{0,\alpha}$. Thus, there is an advantage at looking at this class of functions. Consider what the control on these functions looks like. For $1 > |x - y| > 0$ and $\alpha \in (0, 1)$, we have $|x - y| < |x - y|^\alpha$, and so

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} < \frac{|f(x) - f(y)|}{|x - y|}.$$

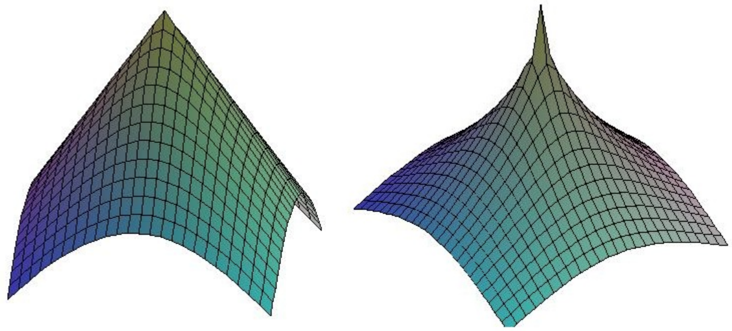


FIGURE 1: Graph of $f(x) = |x|^\alpha$ for $\alpha = 1$ (left) and $\alpha = \frac{1}{2}$ (right).

In particular, this makes it clear that $C^1 \subset C^{0,\alpha}$. Taylor's theorem gives us the error on first order approximation, so one can interpret the Hölder norm as a sort of error estimate. However, it gives much more local information about the function than the L^p norm. Indeed, because integrals do not detect sets of measure 0, and the L^p norm does not tell us about any local behaviour of a function, we would not be able to bound $|D^2 u|$ without additional information.

Now consider the definition of density. Provided the limit exists, the metric density of a measurable set E at a point x is defined as

$$\lim_{r\to 0} \frac{m(B(x,r)\cap E)}{m(B(x,r))}.$$

With this definition in mind, consider the geometry of Hölder space. In his paper, Wang uses the following statements for motivation:

Lemma 1. *If u is a solution of (4) and h is a continuous harmonic function with $h = u$ on ∂B_1, then*

$$|u(x) - h(x)| \le \frac{1}{2n}|1 - |x|^2)\|f\|_\infty. \tag{2}$$

Corollary 1. *For any $0 < \alpha < 1$, there are positive universal contsants $r_0 < 1$ and $\varepsilon_0 > 0$ such that for a solution u of (4) in B_1, with $|u| \le 1$ and $|f| \le \varepsilon_0$, there is a constant A such that*

$$|u(x) - A| \le r_0^\alpha$$

for all $x \in B_{r_0}$. Taking $B_1 = B_{r_0}$, we can iterate this to obtain

$$|u(x) - A_k| \le r_0^{k\alpha}$$

for $x \in B_{r_0^k}$. In other words, we can get very precise local precision of u.

This will be the guiding idea for the measure-theoretic bounds on the sizes of sets.

1.2 Control on sets

Equation (2) tells us that the function u is "almost" harmonic. In other words, we should be able to get some control over it, possibly by looking at averages over balls (we will see this when we introduce the maximal function). Let's look at how we can translate Lemma 1 into set-theoretic language.

We first include the following theorem.

Theorem 1 (Rudin 8.16)**.** *If $f : \Omega \to [0,\infty)$ is a measurable function, μ is a $\sigma-$finite positive measure on the Borel sigma algebra of $\Omega \subset \mathbb{R}^d$, $\varphi : [0,\infty] \to [0,\infty]$ is monotonic, absolutely continuous on $[0,T]$ for every $T < \infty$, $\varphi(0) = 0$ and $\varphi(t) \to \varphi(\infty)$ as $t \to \infty$, then*

$$\int_\Omega (\varphi \circ f) \mathrm{d}\mu = \int_0^\infty \mu\{x : f(x) > t\}\varphi'(t)\mathrm{d}t.$$

Proof. Let E be the set of all $(x,t) \in \Omega \times [0,\infty)$ such that $f(x) > t$. E is clearly a measurable set whenever f is a simple function, and its measurability follows from the approximation of any measurable function by a sequence of increasing simple functions $\phi_1 \leq \phi_2 \leq \ldots \to f$. For every $t \in [0, \infty)$, let E^t be the slice $E^t = \{x : (x,t) \in E\}$. We therefore have

$$\mu(E^t) = \int_\Omega \chi_{\{x: f(x) > t\}}(x,t) \mathrm{d}\mu(x).$$

Hence,

$$\begin{aligned}
\int_0^\infty \mu\{x : f(x) > t\} \varphi'(t) \mathrm{d}t &= \int_0^\infty \int_\Omega \chi_{\{f>t\}}(x,t)\varphi'(t)\mathrm{d}\mu(x)\mathrm{d}t \\
&= \int_\Omega \int_0^\infty \chi_{\{f>t\}}(x,t)\varphi'(t)\mathrm{d}t\mathrm{d}\mu(x) \\
&= \int_\Omega \int_0^{f(x)} \varphi'(t)\mathrm{d}t\mathrm{d}\mu(x) \\
&= \int_\Omega \varphi(f(x))\mathrm{d}\mu(x).
\end{aligned}$$

□

Corollary 2. *We have the relationship*

$$\int_\Omega |u|^p \mathrm{d}x = p \int_0^\infty t^{p-1} m\{x \in \Omega : |u(x)| > t\} \mathrm{d}t.$$

Proof. Apply the previous theorem with $\varphi(t) = t^p$ and $f(x) = u(x)$. We obtain

$$\int_\Omega |u(x)|^p \mathrm{d}x = p \int_0^\infty t^{p-1} \mu\{f > t\} \mathrm{d}t.$$

□

Thus, if $\int_\Omega |u|^p \mathrm{d}x = 1$, we should have that

$$|\{|u| > \lambda\}| \leq \frac{1}{\lambda^p},$$

so that $|\{|u| > \lambda\}|$ is small for λ large. In order to prove that any $u \in L^p$, then, we should start by showing the decay of $|\{|u| > \lambda\}|$. We might hope to prove

$$|\{|u| > \lambda_0\}| \leq \varepsilon |\{|u| > 1\}|$$

for some fixed $\varepsilon > 0$ and $\lambda_0 > 0$, and use the scaling

$$\Delta u(rx) = r^2 f(rx)$$

to get

$$|\{|u| > \lambda_0 \lambda\}| \leq \varepsilon |\{|u| > \lambda\}|.$$

Now compare with (1). We might hope to use the above inequality to get some inductive estimate. Based on the fact that $D^2 u \in L^p$ if Δu is, we might guess that

$$|\{|D^2 u| > \lambda_0 \lambda\}| \leq \varepsilon (|\{|D^2 u| > \lambda\}| + |\{|f| \geq \delta_0 \lambda\}|\}.$$

However, the above is not true; the failure is due to the fact that the condition

$$|D^2 u(x_0)| \leq 1$$

is unstable. In order to get the desired bounds, we will have to use the maximal function.

2 Regularity results

We have already seen the following.

Theorem 2 (Evans, 6.3.2 Thm. 4)**.** *For $u \in H^1(\Omega)$ a solution to (4), we have*

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

We use this to obtain the following.

Proposition 1. *If*

$$\begin{cases} \Delta u = f & \in B_1 \\ u = 0 & \in \partial B_1 \end{cases},$$

then

$$\int_{B_1} |D^2 u|^2 \leq C \int_{B_1} |f|^2.$$

Evans motivates this bound as follows: if $\Delta u = f$, we have

$$\begin{aligned} \int f^2 &= \int (\Delta u)^2 \\ &= \sum_{i,j=1}^{n} \int u_{x_i x_i} u_{x_j x_j} \\ &= -\sum_{i,j=1}^{n} \int u_{x_i x_i x_j} u_{x_j} \\ &= \sum_{i,j=1}^{n} \int u_{x_i x_j} u_{x_i x_j} \\ &= \int \begin{pmatrix} u_1 & \ldots & u_n \end{pmatrix} \left(\frac{\partial^2}{\partial^i \partial^j} \right) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}. \end{aligned}$$

3 Calderòn-Zygmund decomposition

Though it will not be used here, we briefly present the decomposition. We first present the covering lemma.

Lemma 2 (Calderón-Zygmund)**.** *Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$. There is an open set Ω which is a disjoint union of open cubes and its complement F a closed set such that:*

i. $f(x) \leq \alpha$ *a.e. on* F

ii. $\Omega = \cup_k Q_k$*, where the cubes* Q_k *have disjoint interior and*

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} f \leq 2^n \alpha \tag{3}$$

iii. $m(\Omega) \leq \frac{\|f\|_1}{\alpha}$.

Proof. Divide $\mathbb{R}^n$ using a dyadic grid. Here, we see that there are only two cases

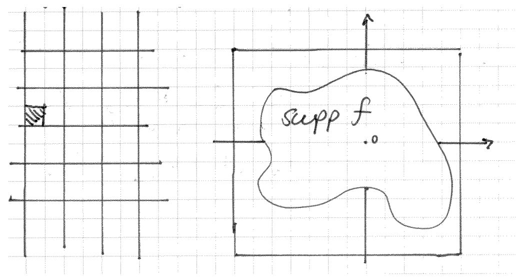
$$\begin{cases} Q_{2^k} \subset Q_{2^n} \\ Q_{2^k} \cap Q^{2^n} = \text{bdry} \end{cases}.$$

Start with a Q_0 sufficiently large so that

$$\frac{1}{|Q_0|} \int_{Q_0} f(x) \mathrm{d}x \leq \alpha$$

for any x, where we are integrating with respect to Lebesgue measure. Let n be the dimension of the space. We get 2^n cubes at each stage of the decomposition.

Stage I: two cases for each new cube



1. $\frac{1}{m(Q)} \int_Q f \leq \alpha$

2. $\frac{1}{m(Q)} \int_Q f > \alpha$.

In the second case, we keep the cube and put it into our collection $\{Q_k\}$, which will eventually become Ω. Note that since each new cube has half the length of the one of the previous generation (say Q_0), we have

$$\frac{1}{m(Q)} \int_Q f \leq \frac{2^n}{m(Q_0)} \int_{Q_0} f \leq 2^n \alpha,$$

so that (3) holds.

In the first case, we keep on dividing and get a countable collection. At each stage, we look again, since almost every point of an L^1 function is a Lebesgue point, we find that $f \leq \alpha$ almost everywhere.

Clearly, then, i. and ii. hold. It remains to see that

$$\begin{aligned} m(\Omega) &= \sum_k m(Q_k) \\ &\leq \sum_k \frac{1}{\alpha} \int_{Q_k} f(x) \mathrm{d}x \\ &= \frac{1}{\alpha} \|f\|_{L^1(\Omega)} \\ &\leq \frac{1}{\alpha} \|f\|_1. \end{aligned}$$

□

We can now show the decomposition of f into "good" and "bad" parts, $f = g + b$. We do so as follows:

$$g(x) = \begin{cases} f(x) & x \in F \\ \frac{1}{m(Q_k)} \int_{Q_k} f(x) \mathrm{d}x & x \in Q_k \subset \Omega \end{cases}.$$

We notice that $g(x) \leq 2^\alpha$ almost everywhere. Then $b = f - g$ and

$$b(x) = \begin{cases} 0 & x \in F \\ f(x) - ⨍ f & x \in Q_k \subset \Omega \end{cases}.$$

Thus, we have that for every cube, $⨍ b = 0$.

4 Vitali covering lemma

The Vitali covering lemma is another simple combinatorial-geometric proof about sets which allows one to consider disjoint unions of balls. It is often used in the proof of the Lebesgue-Radon-Nikodym theorem.

We present the proof with a figure to give intuition, and afterwards we give a modified version of Vitali which Wang uses for his proof. The following proof is from Rudin's Real & Complex analysis.

Theorem 3 (Vitali covering lemma)**.** *Suppose that $\Omega = \cup_{i=1}^n B(x_i, r_i)$ is a finite collection of balls. There is then a set $S = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ such that*

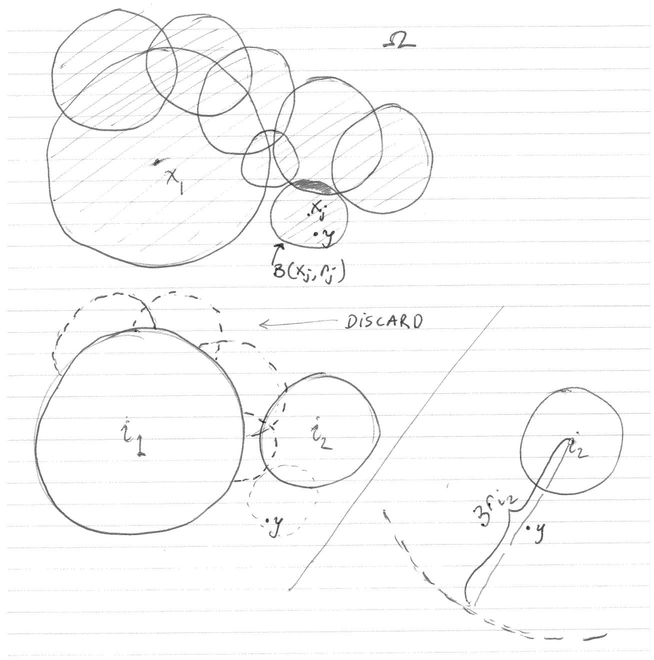


FIGURE 2: Top: original set Ω, a disjoint union of balls. Bottom left: the final result, $S = i_1, i_2$. Bottom right: any y that was in a discarded ball will be within three times the radius of the ball that superceded it.

(i) the balls $B(x_i, r_i)$, $i \in S$ are disjoint,

(ii) $\Omega \subset \cup_{i \in S} B(x_i, 3r_i)$,

(iii) $m(\Omega) \leq 3^k \sum_{i \in S} m(B(x_i, r_i))$.

Proof. The proof is simple. Without loss of generality (re-label if necessary), assume the balls are ordered from largest to smallest radii, i.e. $r_1 \geq r_2 \geq \cdots \geq r_n$. Let $i_1 = 1$, and discard all balls which intersect B_{i_1}. Let i_2 be the largest radius of the remaining balls, and repeat the same procedure. Since there is only a finite numbers of balls, this procedure ends at some i_k. It is obvious that the balls are disjoint, so that (i) holds. Now, suppose that $y \in \Omega$, so $y \in B(x_j, r_j)$ for some i. If $j = i_j$ for some $i_j \in S$, we are done, so suppose the ball $B(x_j, r_j)$ was not chosen. This means it must have intersected some $B(x_i, r_i)$ before it got discarded. Since $r_i \geq r_j$ (otherwise $B(x_i, r_i)$ would've been discarded), we must have $y \in B(x_i, 3r_i)$, which proves (ii). The last statement then follows easily, as

$$(m(B(x, 3r_i))) = 3^k m(B(x, r_i))$$

by the basic properties of Lebesgue measure. $\square$

The modified Vitali is a little more complicated.

Theorem 4 (Modified Vitali)**.** *Let $0 < \varepsilon < 1$ and $C \subset D \subset B_1$ be two measurable sets with $|C| < \varepsilon |B_1|$ and satisfying the following property: for evrey $x \in B_1$ with $|C \cap B_r(x)| \geq \varepsilon |B_r|$, $B_r(x) \cap B_1 \subset D$. Then $|D| \geq (20^n \varepsilon)^{-1} |C|$.*

Proof. First, $|C| < \varepsilon |B_1|$, so for almost all $x \in C$, we can find r_x such that $|C \cap B_{r_x}(x)| = \varepsilon |B_{r_x}|$. To see this, note that if we choose any x in the interior of C, we can certainly choose r_x small enough so that $B_{r_x} \subset C$ and then $|C \cap B_{r_x}(x)| = |B_{r_x}| > \varepsilon |B_{r_x}|$. Moreover, we can choose r_x so big that $|C \cap B_{r_x}(x)| < \varepsilon |B_{r_x}|$. Assuming $|C \cap B_{r_x}(x)|$ varies continuously with r_x, we can invoke the intermediate value theorem to see there is an r_x such that we have equality. Let then $r_x < 2$ be such that $|C \cap B_{r_x}(X)| = \varepsilon |B_{r_x}|$ with

$$|C \cap B_r(x)| < \varepsilon |B_r|$$

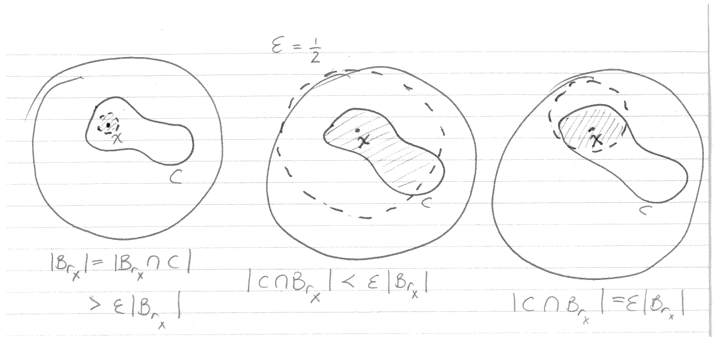


FIGURE 3: Left: too small; center: too big; right: just right!

for all $r_x < r < 2$. The set C can clearly be written as a union of balls, so we can apply the Vitali covering lemma to obtain $x_1, x_2, \dots, x_n \in C$ which satisfy the above condition and such that

$$B_{r_{x_i}} \cap B_{r_{x_j}} = \emptyset, \quad i \neq j, \qquad C \subset \bigcup_{i=1}^{n} B_{5r_{x_i}}(x_i),$$

and since $C \subset B_1$, we also have $C \subset \cup_i B_{5r_{x_i}}(x_i) \cap B_1$. Moreover, since each $B_{r_{x_i}}(x_i)$ satisfies

$$\varepsilon |B_{r_{x_i}}(x_i)| = |C \cap B_{r_{x_i}}(x_i)|,$$

we have

$$\begin{aligned} |C \cap B_{5r_{x_i}}(x_i)| &< \varepsilon |B_{5r_{x_i}}(x_i)| \qquad (5r_i > r_i) \\ &= 5^n \varepsilon |B_{r_{x_i}}(x_i)| \\ &= 5^n |C \cap B_{r_{x_i}}(x_i)|. \end{aligned}$$

Moreover, since $r_{x_i} < 2$, we can say that $|B_{r_{x_i}}(x_i) \leq 4^n |B_{r_{x_i}} \cap B_1|$. Thus,

$$\begin{aligned} |C| &= \left| \bigcup_{i=1}^{n} B_{5r_{x_i}}(x_i) \cap C \right| \\ &\leq \sum_{i=1}^{n} |B_{5r_{x_i}}(x_i) \cap C| \\ &\leq 5^n \sum_{i=1}^{n} \varepsilon |B_{r_{x_i}}(x_i)| \\ &\leq 20^n \sum_{i=1}^{n} \varepsilon |B_{r_{x_i}}(x_i) \cap B_1| \\ &= \varepsilon 20^n \left| \bigcup_{i=1}^{n} B_{r_{x_i}}(x_i) \cap B_1 \right| \\ &= 20^n |D|. \end{aligned}$$

□

5 The Hardy-Littlewood Maximal function

The Hardy-Littlewood function is one of the most important tools in analysis (or so I've been read and been told). In some sense, it gives the 'worst' behiaviour of a function. For $f \in L^p(\mathbb{R}^d)$, the maximal function is defined as

$$(Mf)(x) = \sup_{r>0} \frac{1}{m(B_r)} \int_{B(x,r)} |f| \mathrm{d}m,$$

where $\mathrm{d}m$ denotes the d-dimensional Lebesgue measure.

Remark For $f \in L^1$, Mf is *never* in L^1. Here is a simple example that is easily generalized. Take $d = 1$ and w.l.o.g. suppose $\int_0^1 |f| > 0$ and let $x \geq 2$. By choosing a different radius, we will do worse than the supremum over all radii, and so in particular choosing $r = |x|$ we have

$$\begin{aligned} Mf(x) &= \sup_{0<r<\infty} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x)| \mathrm{d}x \\ &\geq \frac{1}{2|x|} \int_{B_|x|(x)} |f(x)| \mathrm{d}x \\ &\geq \frac{1}{2|x|} \int_0^1 |f(x)| \mathrm{d}x = \frac{C}{|x|}. \end{aligned}$$

Even if we take the maximal function to be 0 on $|x| \leq 2$, the function still diverges due to the factor of $|x|^{-1}$ and therefore cannot be in L^1.

There exist other types of maximal functions. For instance, if μ is any complex Borel measure and m denotes the d-dimensional Lebesgue measure, we first set

$$(P_r\mu)(x) = \frac{\mu(B(x,r))}{m(B(x,r))},$$

and we define the maximal function $M\mu$ for $\mu \geq 0$ by

$$(M\mu)(x) = \sup_{r>0} (P_r\mu)(x).$$

It is easy to show that the function $M\mu$ is lower semicontinuous, hence measurable.

We now use the Vitali covering lemma to prove the following lemma:

Theorem 5. *For a complex Borel measure μ on $\mathbb{R}^d$ and $\lambda > 0$, we have*

$$|\{x \in \mathbb{R}^d : (M\mu)(x) > \lambda\}| \leq 3^d \lambda^{-1} \|u\|,$$

where $\|u\| = |\mu|(\mathbb{R}^d) < \infty$ is the total variation of μ.

Proof. Let μ, λ be given. There is a compact subset K of the open set $\{M\mu > \lambda\}$ (open by the previous claims). For every $x \in K$ and open ball B, we have

$$|\mu|(B) > \lambda m(B).$$

Since K is compact, we can cover K by finitely many balls B, and by the Vitali covering lemma there are balls $B_1, \ldots, B_n$ such that

$$m(K) \leq 3^d \sum_{i=1}^{n} m(B_i) \leq 3^d \lambda^{-1} \sum_{i=1}^{n} |\mu|(B_i) \leq 3^d \lambda^{-1} \|\mu\|$$

where the last inequality follows by disjointness of the balls. The result now follows by taking the supremum over all compact subset $K \subset \{M\mu > \lambda\}$. □

Definition The space **Weak** L^1 is the space of measurable f for which $\lambda m\{|f| > \lambda\} < \infty$ for $0 < \lambda < \infty$.

Remark Any function $f \in L^1$ is in weak L^1. To see this, let $\lambda > 0$, and let $E = \{|f| > \lambda\}$. We then have

$$\lambda m(E) = \int_E \lambda dm \leq \int_E |f| dm \leq \int_{\mathbb{R}^d} |f| dm = \|f\|_1.$$

Taking λ to the other side yields the result.

Corollary 3. *For every $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$,*

$$|\{Mf > \lambda\}| \leq 3^d \lambda^{-1} \|f\|_1.$$

Proof. We can use the previous theorem since for any measurable set $E \subset \mathbb{R}^d$, the mapping

$$E \to \int_E f dm$$

defines a complex Borel measure μ. Writing $\mu = f dm$, we see that the definition of the maximal function for a measure agrees precisely with the Hardy-Littlewood maximal function. □

The above are called weak (1,1) estimates and give à-priori control over Mf in L^1. On the other hand, we have strong (p,p) type estimates, which tell us that if $f \in L^p$ for $p > 1$, then $Mf \in L^p$ as well.

Proof. If $Mf > \lambda$, then by definition, there is a ball of radius r centered at x such that

$$\int_{B(x,r)} |f| dm > \lambda |B(x,r)|.$$

By the Vitali covering lemma, there are disjoint balls $B_1(x_1, r_1), \ldots, B_n(x_n, r_n)$ such that

$$\bigcup_{i=1}^{n} B(x_i, 5r_i) \supset \{x : Mf(x) > \lambda\}.$$

Hence,

$$\begin{aligned}
|\{Mf > \lambda\}| &\leq 3^d \sum_{i=1}^{n} |B_i| \\
&\leq 3^d \lambda^{-1} \int |f| dm.
\end{aligned}$$

Note that this is the same weak estimate we obtained in the previous corollary. We now apply it to obtain strong estimates. Define

$$\beta(x) = \begin{cases} f(x) & |f(x)| > \frac{\lambda}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Clearly then, $\beta(x) \in L^p$ for $f \in L^p$, and we can apply the above weak estimate to β:

$$|\{Mf > \lambda\}| \leq 2 \frac{3^{d}}{\lambda}^{-1} \int_{|f| > \lambda/2} |f| dm.$$

Hence,

$$\begin{aligned}
\|Mf\|_p^p &= \int_{\mathbb{R}^d} \int_0^{Mf(x)} p\lambda^{p-1} \mathrm{d}\lambda \mathrm{d}m \\
&= p \int_0^{Mf(x)} \int_{\mathbb{R}^d} \lambda^{p-1} \mathrm{d}m \mathrm{d}\lambda && \text{(Fubini)} \\
&= p \int_0^{Mf(x)} \int_{\mathbb{R}^d} \lambda^{p-1} m(\{Mf > \lambda\}) \mathrm{d}\lambda \\
&\leq 3^d 2p \int_0^{\infty} \int_{|f| > \lambda/2} \lambda^{p-1} |f| dm \mathrm{d}\lambda \\
&= C_p \|f\|_p^p < \infty.
\end{aligned}$$

□

6 Proof of theorem

We are now in a position to prove the estimate.

Theorem 6 (Calderón-Zygmund). *For u satisfying*

$$\Delta u(x) = f(x), \quad |x| < 2, \tag{4}$$

we have the following estimate for $p \in (1, \infty)$:

$$\int_{B_1} |D^2 u|^p \leq C \left(\int_{B_2} |f|^p + \int_{B_2} |u|^p \right),$$

where D denotes the total derivative operator.

We first prove the following lemma

Lemma 3. *If u is a solution of*

$$\Delta u = f$$

in a domain Ω which contains B_4, the balls of radius 4, and

$$\{M(|f|^2) \leq \delta^2\} \cap \{M|D^2u|^2 \leq 1\} \cap B_1 \neq \emptyset, \tag{5}$$

then there is a constant N_1 so that for any $\varepsilon > 0$, there is a $\delta > 0$ depending only on ε such that

$$|\{M|D^2u|^2 > N_1^2\} \cap B_1| < \varepsilon |B_1|. \tag{6}$$

Proof. Since the set in (5) is not empty, there is $x_0 \in B_1$ such that, for any $r > 0$, we have

$$⨍_{B(r,x_0)} |D^2u|^2 \leq 2^n, \quad ⨍_{B(r,x_0)} |f|^2 \leq 2^n \delta^2. \tag{7}$$

In particular, this allows us to recenter and obtain

$$⨍_{B_4} |D^2u|^2 \leq 1, \quad ⨍_{B_4} |f|^2 \leq \delta^2.$$

Hence (why?)

$$⨍_{B_4} |\nabla u - \overline{\nabla u}_{B_4}|^2 \leq C_1.$$

Letting v be a solution to the corresponding system

$$\begin{cases} \nabla v = 0 & \text{on } B_4 \\ v = u - (\overline{\nabla u})_{B_4} \cdot \mathbf{x} - \overline{u}_{B_4} & \text{on } \partial B_4 \end{cases},$$

we have by minimality of harmonic functions for the Dirichlet energy,

$$\int_{B_4} |\nabla v|^2 \leq \int_{B_4} |\nabla u - \overline{\nabla u}_{B_4}|^2 \leq C_1.$$

Now, by typical regularity results, we have that $v \in C^\infty$, so we can bound the supremum of $D^2 v$:

$$\|D^2 v\|^2_{L^\infty(B_3)} \leq N_0^2.$$

Meanwhile,

$$\int_{B_3} |D^2(u-v)|^2 \leq C \int_{B_4} f^2 \leq C\delta^2.$$

Using the weak L_1 estimate proved in Corollary 3, we have the following bound:

$$\begin{aligned} \lambda |\{x \in B_3 : M|D^2(u-v)| > \lambda\}| &\leq \frac{C}{N_0^2} \int_{B_3} |D^2(u-v)|^2 \\ &\leq \frac{C}{N_0^2} \int_{B_4} f^2 \leq C\delta^2 \end{aligned}$$

by the above. Now, taking $N_1^2 = \max(4N_0^2, 2^n)$, we have that

$$\{x \in B_1 : M|D^2u|^2 > N_1^2\} \subset \{x \in B_1 : M_{B_3}|D^2(u-v)|^2 > N_0^2\}.$$

How to see this? Well, if $y \in B_3$, then certainly

$$\begin{aligned}|D^2u(y)|^2 &= |D^2u(y)|^2 - 2|D^2v(y)|^2 + 2|D^2v(y)|^2 \\ &\leq 2|D^2u(y) - D^2v(y)|^2 + 2N_0^2.\end{aligned}$$

For any $x \in \{x \in B_1 : M_{B_3}|D^2(u-v)|^2 > N_0^2\}$, $r \leq 2$, we have $B_2(x) \subset B_3$ and therefore

$$\sup_{r\leq 2} \fint_{B_r(x)} |D^2u|^2 \leq 2M_{B_3}(|D^2(u-v)|^2)(x) + 2N_0^2 \leq 4N_0^2.$$

On the other hand, for $r > 2$, we have

$$\fint_{B(x,r)} |D^2u|^2 \leq \frac{1}{|B_r|}\int_{B(x_0,2r)} |D^2u|^2 \leq 2^n,$$

since

$$\fint_{B(x_0,r)} |D^2u|^2 \leq 2^n, \quad \fint_{B(x_0,r)} |f|^2 \leq 2^n\delta^2$$

by assumption. Finally, Wang shows that

$$\begin{aligned}|\{x \in B_1 : M(|D^2u|^2) > N_1^2\}| &\leq |\{x \in B_1 : M(|D^2(u-v)|^2 > N_0^2\}| \\ &\leq \frac{C}{N_0^2}\int f^2 \leq C\delta^2/N_0^2 < C\delta^2 = \varepsilon|B_1|\end{aligned}$$

where $\delta^2 = \frac{\varepsilon|B_1|}{C}$ by choice. □

Corollary 4. *Suppose u is a solution of (4) in a domain Ω which contains the ball $B(x,4r)$ for some x, r. Let $B = (x,r)$. If*

$$|\{M(|D^2u|^2) > N_1^2\} \cap B| \geq \varepsilon|B|$$

then

$$B \subset \{M(|D^2u|^2) > 1\} \cup \{M(|D^2u|^2 > \delta^2\}.$$

Proof. By the contrapositive, if

$$|\{M|D^2u|^2 > N_1^2\} \cap B_1| \geq \varepsilon|B_1|$$

then the assumption

$$\{M(|f|^2) \leq \delta^2\} \cap \{M|D^2u|^2 \leq 1\} \cap B_1 \neq \emptyset$$

must be false. In other words, the intersection is empty, and so the original set is entirely contained in the set where $M(|f|^2) > \delta^2$ and/or $M(|D^2u|^2) > 1$. □

From Wang: "The moral of Corollary 2 is that the set $\{x : M(|D^2u|^2) > 1\}$ is bigger than the set $\{x : M(|D^2u|^2) > N_1^2\}$ modulo $\{M(f^2) > \delta^2\}$ if $|\{x : M(|D^2u|^2) > N_1^2\} \cap B| = \varepsilon|B|$. As in the Vitali lemma, we will cover a good portion of the set $\{x : M(|D^2u|^2) > N_1^2\}$ by disjoint balls so that in each of the balls, the density is ε. "

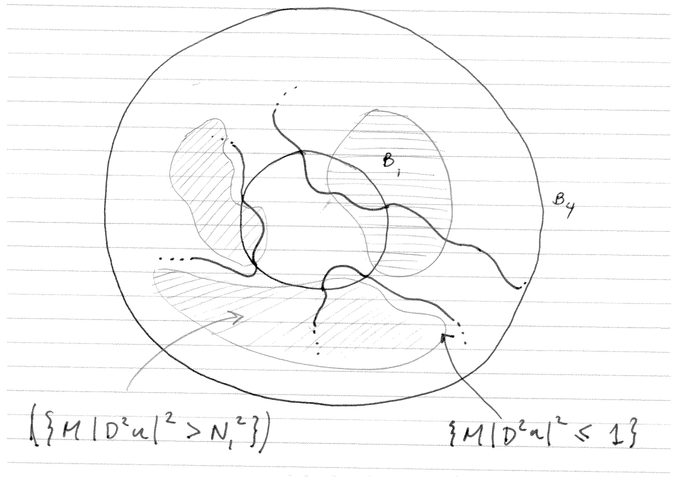
Corollary 5. *Assuming that u is a solution of (4) in $\Omega \supset B_4$, with*

$$\{x \in B_1 : M(|D^2u|^2) > N_1^2\}| \leq \varepsilon|B_1|,$$

then for $\varepsilon_1 = 20^n\varepsilon$, we have

i.

$$|\{x \in B_1 : M(|D^2u|^2) > N_1^2\}| \leq \varepsilon_1(|\{x \in B_1 : M(|D^2u|^2) > 1\}| + |\{x \in B_1 : M(|f|^2) > \delta^2\}|).$$



ii.

$$|\{x \in B_1 : M(|D^2u|^2) > N_1^2\}| \le \varepsilon_1(|\{x \in B_1 : M(|D^2u|^2) > \lambda^2\}| + |\{x \in B_1 : M(|f|^2) > \lambda^2\delta^2\}|)$$

iii.

$$|\{x \in B_1 : M(|D^2u|^2) > (N_1^2)^i\}| \le \sum_{i=1}^{k} \varepsilon_1^i |\{x \in B_1 : M(|f|^2) > \delta^2(N_1^2)^{k-i}\}| + \varepsilon_1^k |\{x \in B_1 : M(|f|^2) > 1\}|.$$

Proof. The important thing is to notice the resemblance between this and how we dealt with the geometry of functions. The proof is straightforward from what we have proven. First, set

$$C = \{x \in B_1 : M(|D^2u|^2) > N_1^2\}$$
$$D = \{x \in B_1 : M(|D^2u|^2) > 1\} \cup \{x \in B_1 : M(|f|^2) > \delta^2\}.$$

Now, we have that $|C| \le \varepsilon|B_1|$ by assumption. To see that the second condition for the modified Vitali lemma is satisfied, we need to show that $|B_r \cap C| \ge \varepsilon|B_r|$ implies $B_r \cap C \subset D$. This follows immediately from the main lemma 3 that

$$|B_r \cap C| \ge \varepsilon|B_r| \Rightarrow D^c \cap B_r = \emptyset.$$

To see ii., we need only iterate the above result on the equation

$$\Delta(\lambda^{-1}u) = \lambda^{-1}f.$$

Here, we are using the scaling property of the Laplacian: $\Delta u(rx) = r^2 f(rx)$. Finally, repeating the same thing over and over on $\lambda = N_1, N_1^2, \dots$ we get further local information on the geometry of these sets. $\square$

We are now in a position to prove the estimate. We do so for $p > 2$.

Theorem 7. *If $\Delta u = f$ in B_4, then*

$$\int_{B_1} |D^2u|^p \le C \int_{B_4} |f|^p + |u|^p.$$

Proof. The first step of the proof is to assume that $\|f\|_p$ is small so that we can obtain

$$\{x \in B_1 : M|D^2u|^2 > N_1^2\}| \le \varepsilon|B_1|.$$

Indeed, this is the crucial bound that makes everything work. We can always obtain this, however, by multiplying f by a small constant and then pulling it out into the larger one.

The trick is to write

$$|D^2u|^p = (|D^2u|^2)^{p/2}$$

and show that

$$M(|D^2u|^2) \in L^{p/2}(B_1)$$

, so that $D^2u \in L^p(B_1)$. Suppose then that $\|f\|_p = \delta$. We then have that (why?)

$$\sum_{i=1}^{\infty}(N_1)^{ip}|\{M(|f|^2) > \delta^2(N_1)^{2i}\}| \leq \frac{pN^p}{\delta^p(N-1)}\|f\|_p^p \leq C.$$

Hence,

$$\begin{aligned}
\int_{B_1}|D^2u|^p \leq \int_{B_1}&(M(|D^2u|^))^{p/2}\mathrm{d}x \\
&= p\int_0^{\infty}\lambda^{p-1}|\{x \in B_1 : M|D^2u|^2 \geq \lambda^2\}|\mathrm{d}\lambda \\
&\leq p\left(|B_1| + \sum_{k=1}^{\infty}(N_1)^{kp}|\{x \in B_1 : M|D^2u|^2 > (N_1)^{2k}\}|\right) \\
&\leq p\left(|B_1| + \sum_{i=1}^{\infty}N_1^{kp}\sum_{i=1}^{k}\varepsilon_1^i|\{x \in B_1 : M|f|^2 \geq \delta^2N_1^{2(k-i)}\}| + \sum_{k=1}^{\infty}N_1^{kp}\varepsilon^k|\{x : M(|D^2u|^2) \geq 1\}|\right) \\
&\leq \sum_{i=1}^{\infty}(N_1)^{ip}|\{M(|f|^2) > \delta^2(N_1)^{2i}\}| \leq C.
\end{aligned}$$

Taking ε_1 so that $N_1^p\varepsilon_1 < 1$, the sum converges and thus the theorem is proved. □

References

[1] Lihe Wang, *A geometric approach to the Calderón-Zygmund estimates*. University of Iowa, `www.math.uiowa.edu/ lwang/cccalderon.pdf`.