The Uniformisation Theorem

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1. Introduction

The goal of this essay is to provide a proof of the uniformisation theorem. We closely follow the approach taken in the last chapter of T. Gamelin's *Complex Analysis* [3]. The main change lies in the organisation of the material, since propositions that are not required for the proof of the uniformisation theorem are omitted.

The Riemann mapping theorem states that any non-empty simply connected proper open subset of the complex plane is conformally equivalent to the open unit disc. The uniformisation theorem is a significant generalisation of the Riemann mapping theorem. It states that every simply connected Riemann surface is conformally equivalent to the open unit disc, the complex plane, or the Riemann sphere. The theorem was originally conjectured by Felix Klein in 1882. It is said [1] that the theorem occurred to him at 2:30am while in the middle of an asthma attack. Rigorous proofs were given by Paul Koebe and Henri Poincare in 1907.

The proof given here is centred around solving Laplace's equation by using Perron's method and Green's functions. The prerequisites are an undergraduate course in complex analysis and point set topology. We assume the Riemann mapping theorem. The proof outline is as follows:

(1) Define harmonic and subharmonic functions on Riemann surfaces.

(2) Define a Perron family of subharmonic functions on a Riemann surface.

(3) Define a Green's function on Riemann surface by taking the supremum of a certain Perron family.

(4) Show that if a Green's function exists on a Riemann surface R, then R can be conformally mapped to the open unit disc via the Riemann mapping theorem.

(5) Define a bipolar Green's function on a Riemann surface, and show that bipolar Green's functions exist for all Riemann surfaces.

(6) Use bipolar Green's functions to show that if a Riemann surface R does not admit a Green's function, then R can be conformally mapped to either the complex plane or the Riemann sphere.

2. Review of Complex Analysis

Before launching into Riemann surfaces, we review some facts from undergraduate complex analysis that will be used in the proof of the uniformisation theorem. No proofs will be provided in this section.

A *domain* $\Omega \subseteq \mathbb{C}$ is a connected open set. Recall that a function $u(x,y) : \Omega \to \mathbb{R}$ is called *harmonic* if its first and second order partial derivatives exist and are continuous, and u satisfies Laplace's equation: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Example 2.1. *It is a straight-forward calculation to check that $f(z) = \log|z|$ is a harmonic function.*

Proposition 2.1. (Poisson Integral) *Let $h(\theta)$ be a continuous function defined on the boundary of an open disc D of radius $a > 0$. Then the function $u : D \to \mathbb{R}$ given by*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \varphi) + r^2}d\varphi$$

is a harmonic function, and u extends to a continuous function on $\overline{D}$ that agrees with h on ∂D.

Definition 2.1. *Let Ω be a domain in the complex plane. A continuous function $u : \Omega \to [-\infty, \infty)$ is* **subharmonic** *if for each $z_0 \in \Omega$, there exists an $\varepsilon > 0$ such that*

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

for all $r \in (0, \varepsilon)$.

By evaluating the Poisson integral formula at the centre of a disc (after possibly translating the origin), we see that every harmonic function is subharmonic.

Proposition 2.2. (Maximum Principle) *Let u be a subharmonic function on a domain Ω. If u attains a maximum at $p \in \Omega$, then $u \equiv const$.*

Proposition 2.3. *Let u be a continuous function on a domain Ω. Then u is subharmonic if and only if for all discs $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ such that $\overline{B_r(z_0)} \subset \Omega$, and for all harmonic functions $v : \overline{B_r(z_0)} \to \mathbb{R}$, if $u \leq v$ on ∂B then $u \leq v$ on all of B.*

Proposition 2.4. *If $f : \Omega \to \mathbb{C}$ is a holomorphic function defined on a domain Ω, then $\log |f(z)|$ is a subharmonic function.*

The Cauchy-Riemann equations yield a connection between harmonic functions and holomorphic functions, as shown by the two following propositions.

Proposition 2.5. *Let V, U be open sets in $\mathbb{C}$. If $F : V \to U$ is a holomorphic function and $u : U \to \mathbb{R}$ is harmonic, then $u \circ F$ is harmonic.*

Before stating the next proposition, we must introduce some terminology. The same terminology will be used for Riemann surfaces so we shall use definitions valid in general topological spaces.

Let R be a topological space. A *path* in R from $x_0 \in R$ to $x_1 \in R$ is a continuous map $f : [0,1] \to R$ such that $f(0) = x_0$ and $f(1) = x_1$. We define a notion of equivalence between two paths with the same endpoints if one can be "dragged" onto the other while pinning down the endpoints. More precisely, two paths $f : [0,1] \to R$ and $g : [0,1] \to R$ such that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$ are *path homotopic* if there exists a continuous map $F : [0,1] \times [0,1] \to R$ such that

(1) $F(t, 0) = f(t)$ and $F(t, 1) = g(t)$ for all $t \in [0, 1]$, and
(2) $F(0, s) = x_0$ and $F(1, s) = x_1$ for all $s \in [0, 1]$.

A topological space R is *path connected* if for any two distinct points $x_0, x_1 \in R$, there is a path from x_0 to x_1. A space R is *simply connected* if it is path connected, and any two paths with the same starting and ending points are path homotopic. For the case when R is a domain in $\mathbb{C}$, we can intuitively think of R as simply connected when it has no holes.

Proposition 2.6. *Let u be a real-valued harmonic function on a simply connected domain $\Omega \subseteq \mathbb{C}$. Then there exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that $Re(F) = u$. Furthermore, the imaginary part of this function is uniquely determined up to an additive real constant.*

Certain sequences of harmonic functions behave nicely and allow us to extract a subsequence that converges to a harmonic function.

Proposition 2.7. *Let $\{u_n\}$ be a sequence of harmonic functions on Ω that is uniformly bounded on each compact subset of Ω. Then $\{u_n\}$ has a convergent subsequence that converges uniformly to a harmonic function on each compact subset of Ω.*

Proposition 2.8. (Harnack's Inequality) *Let K be a compact subset of a domain Ω, and u a non-negative harmonic function. There is a constant $C > 1$ such that*

$$\frac{1}{C} \leq \frac{u(x)}{u(y)} \leq C$$

for all $x, y \in K$.

Proposition 2.9. (Harnack's Principle) *Let $\{u_n\}$ be a pointwise increasing sequence of harmonic function on a domain Ω. Then either $u_n(x) \to \infty$ for all $x \in \Omega$, or $\{u_n\}$ converges uniformly on compact subsets of Ω to a harmonic function.*

Next, we state the monodromy theorem. This theorem allows us under certain conditions to extend an analytic function from a subset of a simply connected domain Ω to all of Ω.

Theorem 2.1. (Monodromy Theorem) *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. Suppose $f : \Omega \to \mathbb{C}$ is analytic at z_0, and it is possible to analytically extend f on any curve starting at z_0. Then for two paths γ_1 and γ_2 from z_0 to z_1, the analytic continuation of f along γ_1 and γ_2 yield the same value at z_1.*

Definition 2.2. *Let U, V be open sets in $\mathbb{C}$. A bijective holomorphic function $f : U \to V$ is called* **conformal**.

It turns out that if a bijective function is holomorphic, then its inverse is automatically holomorphic. For this reason, conformal maps between domains in the complex plane are sometimes called biholomorphic.

Proposition 2.10. *Let U, V be open sets in $\mathbb{C}$, and let $f : U \to V$ be a bijective holomorphic function. Then $f^{-1} : V \to U$ is a holomorphic function.*

To conclude this section, we state the Riemann mapping theorem.

Theorem 2.2. (Riemann Mapping Theorem) *Let Ω be a non-empty, simply connected subset of $\mathbb{C}$ that is not the whole of $\mathbb{C}$. Then there exists a conformal map from Ω to the open unit disc.*

3. Riemann Surfaces

We explore the notion of a Riemann surface. Intuitively, it can be thought as a globally curved version of the complex plane which, on a small enough scale, resembles a patch of the complex plane.

First, we establish some terminology. Let R be a topological space. A *chart* (U, φ) is an open set U in R together with a homeomorphism φ from U to an open set $\varphi(U) \subseteq \mathbb{C}$.

Two charts (U, φ), (V, ψ) are *compatible* if either $U \cap V = \emptyset$, or if $U \cap V \neq \emptyset$ and the *transition map* $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is analytic.

A collection of charts $(U_\alpha, \varphi_\alpha)$ such that $\{U_\alpha\}$ forms an open covering of R and any two charts are compatible is called an *atlas*. An atlas is called *maximal* if it is not contained in any strictly larger atlas. In other words, given a chart in a maximal atlas, any other chart that is compatible with the original chart is in the maximal atlas.

It is usually not convenient to explicitly describe a maximal atlas. However, it turns out that each smooth atlas for R is contained in a unique maximal atlas. This fact can be proved by considering the set of all charts that are compatible with the original atlas and showing that this set is indeed an atlas. Therefore, by specifying some atlas for R, we uniquely determine the maximal atlas containing it.

Recall that a topological space R is *Hausdorff* if for any points $p, q \in R$, there exists disjoint open sets U, V such that $p \in U$ and $q \in V$. The topological space R is *second countable* if there is a countable basis for its topology.

We can now give a precise definition of a Riemann surface.

Definition 3.1. *A* **Riemann surface** *is a connected, Hausdorff, second countable topological space R with a maximal atlas.*

Example 3.1. *The complex plane $\mathbb{C}$ is a Riemann surface with an atlas consisting of a single chart (U, φ), where $U = \mathbb{C}$ and $\varphi = id$. Similarly, any domain in the complex plane is a Riemann surface.*

Example 3.2. *The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the simplest nontrivial Riemann surface. We can use the charts (U_1, φ_1) and (U_2, φ_2) as our atlas, where $U_1 = \mathbb{C}$, $\varphi_1(z) = z$ and $U_2 = \hat{\mathbb{C}} \backslash \{0\}$, $\varphi_2(z) = 1/z$. The transition map is indeed analytic: $\varphi_1 \circ \varphi_2^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}$, $\varphi_1 \circ \varphi_2^{-1}(z) = 1/z$.*

Let $(U_\alpha, \varphi_\alpha)$ be a chart on R. If $\overline{\mathbb{D}} \subset \mathbb{C}$ is a closed unit disc contained in $\varphi_\alpha(U_\alpha)$, then we let $\Delta = \varphi_\alpha^{-1}(\mathbb{D})$ and call (Δ, φ_α) a *coordinate disc* in R. Note that φ_α^{-1} is well-defined on $\partial\mathbb{D}$.

We can see that for any $q \in R$, there is a coordinate disc (Δ, φ) containing q such that $\varphi(q) = 0$. There is a chart (V, ψ) such that $q \in V$, and since $\psi(V)$ is open, there is a disc of radius $r > 0$ centred at $\psi(q)$ such that its closure is contained in $\psi(V)$. Call this disc B_r and let $\Delta = \psi^{-1}(B_r)$. Then we can translate and rescale to obtain a map $\varphi : \Delta \to \mathbb{D}$ where $\varphi(p) = (\psi(p) - \psi(q))r^{-1}$.

A map $f : R \to S$ between Riemann surfaces R, S, is *analytic* at $p \in R$ if there is a coordinate disc (Δ_1, φ_1) in R containing p and a coordinate disc (Δ_2, φ_2) in S containing $f(p)$ such that $\varphi_2 \circ f \circ \varphi_1^{-1}$ is analytic at $\varphi_1(p)$. We say f is *analytic* if f is analytic at every $p \in R$. In the specific case $f : R \to \mathbb{C} \cup \{\infty\}$, we say f *meromorphic* at $p \in R$ if f is analytic at p.

Two Riemann surfaces R, S are *conformally equivalent* if there is a bijective analytic map $f : R \to S$.

Proposition 3.1. *Every Riemann surface R has a countable basis of precompact coordinate discs.*

Proof. We only sketch the proof. For a more detailed proof, we refer the reader to [4]. Each point in R is contained in a chart $(U_\alpha, \varphi_\alpha)$, and we can use these charts to form an open cover $\{U_\alpha\}$ of R. Since R is second countable, we can obtain a countable subcover $\{U_i\}$. For each (U_i, φ_i), obtain a countable basis of precompact discs in the image $\varphi_i(U_i)$ by considering balls of rational centre and radius. Pulling this basis back to R for each chart (U_i, φ_i) yields a countable basis of precompact coordinate discs.

□

Definition 3.2. *Let R be a Riemann surface. A real-valued function $u : R \to \mathbb{R}$ is harmonic at $p \in R$ if there exists a coordinate disc (Δ, φ) containing p such that $u \circ \varphi^{-1} : \mathbb{D} \to \mathbb{R}$ is a harmonic function. If u is harmonic at each $p \in R$, we say u is a* **harmonic function** *on R.*

We note that harmonicity at p is invariant with respect to the choice of coordinate disc. Indeed, if (Δ_1, φ_1) and (Δ_2, φ_2) are two coordinate discs containing p and $u \circ \varphi_1^{-1}$ is harmonic, then $u \circ \varphi_2^{-1} = (u \circ \varphi_1^{-1}) \circ (\varphi_1 \circ \varphi_2^{-1})$ is the composition of a harmonic function with a holomorphic function, hence is harmonic.

Definition 3.3. *A continuous function $u : R \to [-\infty, \infty)$ is* **subharmonic** *if for every coordinate disc (Δ, φ), if $v : \overline{\Delta} \to \mathbb{R}$ is a harmonic function such that $u(p) \leq v(p)$ for all $p \in \partial\Delta$, then $u(p) \leq v(p)$ for all $p \in \Delta$.*

Equivalently, we can define a continuous function $u : R \to [-\infty, \infty)$ to be subharmonic if for each $p \in R$, there exists a coordinate disc (Δ, φ) containing p such that $u \circ \varphi^{-1} : \mathbb{D} \to \mathbb{R}$ is a subharmonic function. As expected, every harmonic function on R is subharmonic on R.

Proposition 3.2. (Maximum Principle) *Let u be a subharmonic function on a Riemann surface R. If u attains a maximum at $p \in R$, then $u \equiv const.$*

Proof. Let M be the supremum of u. Consider the set $\Gamma = \{p \in R : u(p) = M\}$. Then $\Gamma = u^{-1}(\{M\})$ is closed by continuity of u. Let $q \in \Gamma$. There is a coordinate disc (Δ, φ) containing q such that $u \circ \varphi^{-1} : \mathbb{D} \to \mathbb{R}$ is a subharmonic function. If we let $z_0 = \varphi(q)$, we see that $u \circ \varphi^{-1}$ is a subharmonic function on $\mathbb{D}$ that achieves its maximum at $z_o \in \mathbb{D}$. By the maximum principle for subharmonic functions on domains in the complex plane, $u \circ \varphi^{-1} \equiv const.$ Since φ is bijective, $u \equiv M$ in Δ. This shows that Γ is open. Since Γ is non-empty, open, and closed, by connectedness of R we have $R = \Gamma$. □

Let (Δ, φ) be a coordinate disc, and u be a subharmonic function on R. By using the Poisson integral, we can solve the Dirichlet problem

$$\begin{cases} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 & \text{in } \mathbb{D} \\ w = u \circ \varphi^{-1} & \text{on } \partial\mathbb{D}. \end{cases}$$

We then define

$$u_\Delta(p) = \begin{cases} u(p) & p \notin \Delta \\ w \circ \varphi(p) & p \in \Delta. \end{cases}$$

By construction, u_Δ is harmonic in Δ. Notice that $u_\Delta \geq u$ and that the function u_Δ is subharmonic.

It is immediate that if u, v are subharmonic functions, then $\max\{u, v\}$ is a subharmonic function. Therefore, the following definition makes sense:

Definition 3.4. *A* **Perron family** *on R is a collection $\mathcal{F}$ of subharmonic functions such that*
(1) If $u_1, u_2 \in \mathcal{F}$, then $\max\{u_1, u_2\} \in \mathcal{F}$.
(2) If $u \in \mathcal{F}$, then $u_\Delta \in \mathcal{F}$.

Taking the supremum of a Perron family is a useful method of obtaining a harmonic function, and will be used in our definition of the Green's function.

Proposition 3.3. *Let $\mathcal{F}$ be a Perron family on R. Then $u(p) = \sup\{v(p) : v \in \mathcal{F}\}$ is either harmonic, or $u(p) = +\infty$ for all $p \in R$.*

Proof. We proceed as in [2]. First, we show that the set $\{z \in R : u(z) < \infty\}$ is open and u is harmonic in this set.

Let $p \in R$ be such that $u(p) < \infty$ and let (Δ, φ) be a coordinate disc containing p. There exists a sequence $\{f'_n\} \in \mathcal{F}$ such that $f'_n(p)$ converges to $u(p)$. If we define $f_n = \max\{f'_1, \ldots, f'_n\}$, we obtain a sequence of increasing subharmonic functions. Therefore, $\{(f_n)_\Delta\}$ is a sequence of increasing functions that are harmonic on Δ.

By Harnack's Theorem, $f = \sup\{(f_n)_\Delta\}$ is either $+\infty$ or harmonic in Δ. But since $(f_n)_\Delta \in \mathcal{F}$, we have $(f_n)_\Delta \leq u$, hence $f \leq u$. Therefore $f(p) < \infty$, and so f is harmonic in Δ. Note that $f(p) = u(p)$. Indeed, $f_n(p)$ is an increasing sequence converging to $u(p)$, so $f(p) = \sup_n\{(f_n)_\Delta(p)\} \geq \sup_n\{f_n(p)\} = u(p)$.

We now show that $f = u$ in Δ, and hence $u < \infty$ in Δ and u is harmonic in Δ. Suppose $f(q) < u(q)$ for some $q \in \Delta$. There exists a sequence $\{g'_n\} \in \mathcal{F}$ such that $g'_n(q)$ converges to $u(q)$. If we define $g_n = \max\{g'_1, \ldots, g'_n, f'_1, \ldots, f'_n\}$, we obtain a sequence of increasing subharmonic functions such that $g_n \geq f_n$. Proceeding as above, we obtain $g = \sup\{(g_n)_\Delta\}$, a harmonic function in Δ such that $g(p) = u(p)$, $g(q) = u(q)$, and $g \geq f$. But since $g(p) = f(p)$ and $g(q) > f(q)$, then $(f - g)$ is a non-constant harmonic function on Δ that achieves its maximum at p. Hence we have a contradiction to the maximum principle.

We have shown that the set $\{z \in R : u(z) < \infty\}$ is open and u is harmonic in this set. Its complement, $\{z \in R : u(z) = \infty\}$ is also open. If $u(p) = \infty$, then using the above notation, $f = \infty$ in Δ by Harnack's theorem. But $f \leq u$ in Δ, so $u = \infty$ in Δ.

Therefore, $\{z \in R : u(z) < \infty\}$ is both open and closed, hence is either equal to R or $\emptyset$. If it is equal to R, then u is harmonic, else $u = \infty$.

$\square$

4. Green's Functions

The goal of this section is to develop the machinery of Green's functions on a Riemann surface.

Fix a point $q \in R$, and let (Δ, φ) be a coordinate disc containing q such that $\varphi(q) = 0$. Let $\mathcal{P}_q$ be a family of subharmonic functions on $R\backslash\{q\}$ such that

(1) every $u \in \mathcal{P}_q$ has compact support, and
(2) every $u \in \mathcal{P}_q$ is such that $v(p) = u(p) + \log|\varphi(p)|$ is subharmonic on Δ.

This definition makes $\mathcal{P}_q$ a Perron family on $R\backslash\{q\}$. This is a non-empty set since $u = 0 \in \mathcal{P}_q$; this follows since $\log|\varphi(p)|$ is a subharmonic function by Proposition 2.4.

Definition 4.1. *Suppose* $\sup\{u(p) : u \in \mathcal{P}_q\} < \infty$ *for some* $p \in R$*. A* **Green's function** *for* R *with pole at* q *is defined as* $g(p, q) = \sup\{u(p) : u \in \mathcal{P}_q\}$ *for all* $p \in R\backslash\{q\}$*.*

If the supremum is identically infinite, then we say that Green's function for R with pole at q does not exist. The Green's function will be a central tool in the proof of the uniformisation theorem. We collect some more important facts about the Green's function.

Proposition 4.1. *Let* $g(p, q)$ *be a Green's function for* R *with pole at* q*. Then* $g(p, q) > 0$ *and* $g(p, q)$ *is harmonic for all* $p \in R\backslash\{q\}$*. Furthermore, if* (Δ, φ) *is a coordinate disc such that* $\varphi(q) = 0$*, then* $h(p) = g(p, q) + \log|\varphi(p)|$ *is harmonic on* Δ*.*

Proof. We can see that $g(p, q)$ is harmonic in $R\backslash\{q\}$ by Proposition 3.3. We have $g(p, q) \geq 0$ since $u = 0 \in \mathcal{P}_q$. By the maximum principle, $-g$ cannot attain zero on $R\backslash\{q\}$, from which it follows that $g(p, q) > 0$ for $p \in R\backslash\{q\}$.

Next, we want to show that $h(p) : \Delta \to \mathbb{R}$, $h(p) = g(p, q) + \log|\varphi(p)|$ is harmonic. We can see that $h(p)$ is harmonic on $\Delta\backslash\{q\}$, since

$$h \circ \varphi^{-1}(z) = g \circ \varphi^{-1}(z) + \log|z|$$

is a harmonic function on the punctured unit disc. If we can show that h is bounded, Riemann's removable singularity theorem will allow us to conclude that h is harmonic on Δ.

Let M denote the maximum achieved by $g(p, q)$ on the compact set $\partial\Delta = \{p \in R : |\varphi(p)| = 1\}$. For any $u \in \mathcal{P}_q$, we have $u \leq M$ on $\partial\Delta$. Since $u + \log|\varphi(p)|$ is subharmonic on Δ and $\log|\varphi(p)| = 0$ on $\partial\Delta$, we can apply the maximum principle to conclude $u(p) + \log|\varphi(p)| \leq M$ for all $p \in \Delta$. Taking the supremum, we obtain

$$h(p) = g(p, q) + \log|\varphi(p)| \leq M.$$

On the other hand, the function $v : R \to \mathbb{R}$ defined as

$$v(p) = \begin{cases} -\log|\varphi(p)| & p \in \Delta \\ 0 & p \notin \Delta. \end{cases}$$

is a member of $\mathcal{P}_q$.

Hence $g(p, q) \geq -\log|\varphi(p)|$ for all $p \in \Delta$. Therefore we have shown

$$0 \leq h(p) \leq M.$$

This completes the proof.

□

Proposition 4.2. *Let $g(p, q)$ be a Green's function on R with a pole at q. Then*

$$\inf_{p \in R} g(p, q) = 0.$$

Proof. Select a coordinate disc (Δ, φ) containing q such that $\varphi(q) = 0$. Suppose $\inf_{p\in R} g(p, q) = a > 0$. Then $g(p, q) - a > 0$ for all $p \in R \backslash \{q\}$, and furthermore $g(p, q) - a + \log|\varphi(p)|$ is harmonic for $p \in \Delta$.

Let $u \in \mathcal{P}_q$. By definition, $u(p) + \log|\varphi(p)|$ is subharmonic in Δ. The logarithms cancel after subtracting $g(p, q) - a + \log|\varphi(p)|$ from $u(p) + \log|\varphi(p)|$, and we obtain a subharmonic function $u(p) - (g(p, q) - a)$ on Δ. Since u is compactly supported, $u(p) - (g(p, q) - a) < 0$ outside a compact set, and we can apply the maximum principle to conclude $u(p) < g(p, q) - a$ for all $p \in R$. Taking the supremum over all such $u \in \mathcal{P}_q$, we obtain $g(p, q) \leq g(p, q) - a$, a contradiction.

□

We now prove a surprising fact about the existence of Green's functions.

Proposition 4.3. *Suppose a Green's function $g(p, q_0)$ with pole at q_0 exists for some $q_0 \in R$. Then a Green's function with pole at q exists for all points $q \in R$.*

Before launching into the proof, we lay some ground work in preparation. Fix $q \in R$, and let (U, φ) be a chart containing q such that $\varphi(q) = 0$. Choose $r > 0$ small enough such that

$$\Delta_r = \{p \in R : |\varphi(p)| \leq r\} \subset U$$

is defined. We define the following Perron family of functions:

$$\mathcal{F} = \{v : R \backslash \Delta_r \to \mathbb{R} : v \text{ subharmonic},\ v \leq 1,\ v \text{ compactly supported}\}.$$

We know that $\tilde{v} : R\backslash\Delta_r \to \mathbb{R}$ defined by $\tilde{v}(p) = \sup\{v(p) : v \in \mathcal{F}\}$ is a harmonic function by Proposition 3.3. The analysis of $\tilde{v}$ will be our main tool in the proof.

Let $s > r$ such that $\Delta_s = \{p \in R : |\varphi(p)| \le s\} \subset U$ is defined. By looking for a solution of the form $c_0 + c_1 \log|z|$, we can solve the Dirichlet problem for a function w_0 in the annulus $\{z \in \mathbb{C} : r < |z| < s\}$ with boundary conditions $w_0(z) = 1$ if $|z| = r$ and $w_0(z) = 0$ if $|z| = s$:

$$w_0(z) = 1 - \frac{\log(r)}{\log(r/s)} + \frac{\log|z|}{\log(r/s)}.$$

We compose with our coordinate function to obtain a subharmonic function on $R\backslash\Delta_r$:

$$v_0(p) = \begin{cases} w_0 \circ \varphi(p) & p \in \Delta_s\backslash\Delta_r \\ 0 & p \notin \Delta_s. \end{cases}$$

We see that $v_0 \in \mathcal{F}$, and $v_0(p) \to 1$ as $p \to \partial\Delta_r$ by design. Since $1 \ge \tilde{v} \ge v_0$, we conclude $\tilde{v}(p) \to 1$ as $p \to \partial\Delta_r$.

Since $v = 0 \in \mathcal{F}$, we have $0 \le \tilde{v} \le 1$. But if $\tilde{v}(p) = 0$ for some $p \in R\backslash\Delta_r$, we can apply the maximum principle on $-\tilde{v}$ to conclude that $\tilde{v} \equiv \mathit{const}$. Similarly, $\tilde{v}$ cannot achieve 1 unless it is a constant. Hence we are left with two cases: either $\tilde{v}(p) \in (0,1)$, or $\tilde{v} \equiv 1$. The strategy will be to show that $\tilde{v}(p) \in (0,1)$ if and only if a Green's function $g(p,q)$ with pole at q exists.

Proof. We now prove Proposition 4.3. Let Γ be the set of all $q \in R$ such that a Green's function $g(p,q)$ with pole at q exists. First, we show that Γ is open.

Fix $q_0 \in \Gamma$, and let (U, φ) be a chart containing q_0 such that $\varphi(q_0) = 0$. Define $\Delta_r \subset U$ and $\tilde{v} : R\backslash\Delta_r \to \mathbb{R}$ as above. It will be shown that if $g(p,q_0)$ exists at $q_0 \in R$, then $\tilde{v} \in (0,1)$.

Select $a \in \mathbb{R}$ such that $g(p,q_0) > a > 0$ for all $p \in \partial\Delta_r$. Therefore $\tilde{v}(p) \le 1 < g(p,q_0)/a$ for $p \in \partial\Delta_r$, and hence by the maximum principle,

$$\tilde{v}(p) < g(p,q_0)/a$$

for all $p \in R\backslash\Delta_r$. Since $\inf g(p,q_0) = 0$, we conclude that $\inf \tilde{v}(p) = 0$ and hence $\tilde{v} \not\equiv 1$. Therefore $\tilde{v} \in (0,1)$.

We next show that if $\tilde{v} \in (0,1)$, then $g(p,q)$ exists for all $q \in \Delta_r$. Let $q \in \Delta_r$, and define Δ_s such that $\Delta_r \subset \Delta_s \subset U$ as before. Take any $u \in \mathcal{P}_q$. Then $u(p) + \log|\varphi(p) - \varphi(q)|$ is subharmonic on Δ_s. Choose $C > 0$ large enough such that for all $p \in \partial\Delta_r \cup \partial\Delta_s$ we have $|\log|\varphi(p) - \varphi(q)|| \leq C$. Select $M_u > 0$ such that for all $p \in \partial\Delta_s$ we have $|u(p)| \leq M_u$. Therefore, we have the estimate

$$u(p) + \log|\varphi(p) - \varphi(q)| \leq C + M_u$$

for all $p \in \partial\Delta_s$, and hence by the maximum principle, for all $p \in \Delta_s$. In particular,

$$u(p) \leq 2C + M_u$$

for all $p \in \partial\Delta_r$. Since $\tilde{v} = 1$ on $\partial\Delta_r$, we have $u(p) \leq (2C + M_u)\tilde{v}(p)$ for all $p \in \Delta_r$, and hence by the maximum principle and since u has compact support, the estimate holds for all $p \in R \backslash \Delta_r$. In particular, if we take the maximum over all $p \in \partial\Delta_s$ we obtain

$$M_u \leq (2C + M_u) \max_{p \in \partial\Delta_s} \tilde{v}(p).$$

We can solve for M_u and obtain a bound independent of u, that is, $M_u \leq K$ for some $K > 0$. Then $u \leq K$ on $\partial\Delta_s$, and by the maximum principle, $u \leq K$ on Δ_s. Since this holds for all $u \in \mathcal{P}_q$, by taking the supremum we see that the Green's function exists. This shows that Γ is open.

Let $q_0 \in \Gamma^c$, that is, suppose there does not exists a Green's function $g(p, q_0)$ with pole at q_0. Using the same notation as before, set up a chart (U, φ) centred at q_0, define a disc Δ_r, and obtain the associated function $\tilde{v} : R\backslash\Delta_r \to \mathbb{R}$. If $\tilde{v} \in (0,1)$, then by the above argument $g(p,q)$ exists for all $q \in \Delta_r$, which is a contradiction. Therefore $\tilde{v} \equiv 1$. If a Green's function exists with pole at some $q \in \Delta_r$, then $\tilde{v} \in (0,1)$ by the argument at the beginning of this proof. Therefore, no Green's function exists in Δ_r, and hence Γ^c is open.

Since Γ is both open and closed, either $\Gamma = R$ or $\Gamma = \emptyset$.

□

Proposition 4.4. *Let S be a Riemann surface. Let R be a Riemann surface that is a connected open subset of S. Suppose R has an atlas that consists of a finite number of precompact coordinate discs in S. Then for all $q \in R$, there exists a Green's function on R with pole at q.*

Proof. We only need to show existence of a Green's function with a pole at a single q, so we can select a q that is inside a precompact coordinate disc (U, φ) which is at the "edge" of R, i.e. such that $\partial R \cap \partial U \subset S$ is non-empty. Select a point $b \in \partial R \cap \partial U$. Then $\varphi(b) \in \partial\mathbb{D}$.

Consider the function $w : \mathbb{D} \to \mathbb{R}$ defined by

$$w(z) = \mathrm{Re}(\overline{\varphi(b)} \cdot z) - 1.$$

One can check that w is subharmonic (indeed, it is harmonic), $w(\varphi(b)) = 0$, and $w(z) < 0$ for all other $z \neq \varphi(b)$. We want to extend $\zeta(p) = w \circ \varphi(p)$ to all of R such that it remains subharmonic and $\zeta(p) \leq 0$ with equality if and only if $p = b$. This can be done by extending ζ one coordinate disc at a time. If (V, ψ) is another coordinate disc such that $U \cap V \neq \emptyset$, then $\zeta(p) \leq 0$ for all $p \in \partial V \cap U$. Using Tietze extension theorem, we can continuously extend ζ along ∂V such that $\zeta(p) < 0$ for all $p \in \partial V \backslash U$. Since ζ is now defined along the boundary of the coordinate disc V, we can then use the Poisson formula to solve for a harmonic function defined inside V and use it to define ζ inside V. This may redefine ζ on $U \cap V$, but that is unimportant; ζ will now be defined on $U \cup V$ such that it is subharmonic and $\zeta(p) \leq 0$ with equality if and only if $p = b$. This procedure can be repeated for all of the finitely many coordinate discs which define R.

Now, select an $r > 0$ small enough such that $\Delta_r = \{p \in R : |\varphi(p)| \leq r\} \subset U$ is defined. Define $\mathcal{F}$ and $\tilde{v} : R \backslash \Delta_r \to \mathbb{R}$ as in the previous proof. Select $a > 0$ such that $-\zeta(p) > a$ for all $p \in \partial \Delta_r$. For any $v \in \mathcal{F}$, we have $v(p) + \zeta(p)/a \leq 0$ for all $p \in \partial \Delta_r$. Using subharmonicity, the fact that v has compact support in R, and the maximum principle, we conclude that $v(p) + \zeta(p)/a \leq 0$ for all $p \in R \backslash \Delta_r$. Taking the supremum, we obtain $\tilde{v}(p) \leq -\zeta(p)/a$ on $R \backslash \Delta_r$. Since $-\zeta(p) \to 0$ as $p \to b$, we also have $\tilde{v}(p) \to 0$ as $p \to b$. Therefore $\tilde{v} \not\equiv 1$, and by the proof of the previous theorem, a Green's function for R exists with pole at q for all $q \in \Delta_r$, hence for all $q \in R$.

□

The uniformisation theorem classifies all simply connected Riemann surfaces R. To conclude this section, we show that if a Green's function exists on R, then R is conformally equivalent to the open unit disc. First, we need a lemma to help construct our conformal map to the unit disc.

Lemma 4.1. *Suppose R is a simply connected Riemann surface such that a Green's function $g(p, q)$ with pole at q exists. There exists an analytic function $F : R \to \mathbb{C}$ such that*

$$|F(p)| = e^{-g(p,q)}.$$

Proof. Let $\mathcal{A} = \{(\Delta_\alpha, \varphi_\alpha)\}$ be an atlas of charts for R. For convenience, assume that each chart $(\Delta_\alpha, \varphi_\alpha) \in \mathcal{A}$ is a coordinate disc.

Select a chart $(\Delta_\alpha, \varphi_\alpha) \in \mathcal{A}$. First, suppose $q \notin \Delta_\alpha$. Then $g(p) := g(p, q)$ is harmonic in Δ_α, hence $g \circ \varphi_\alpha^{-1}$ is a real harmonic function on $\mathbb{D}$. Therefore, by Proposition 2.6 there exists an analytic function $G_\alpha : \mathbb{D} \to \mathbb{C}$ such that $g \circ \varphi_\alpha^{-1} = \mathrm{Re}(G_\alpha)$. We define $H_\alpha : \Delta_\alpha \to \mathbb{C}$ to be the analytic function $H_\alpha = G_\alpha \circ \varphi_\alpha$. It follows that

$$\text{Re}(H_\alpha) = \text{Re}(G_\alpha \circ \varphi_\alpha) = \text{Re}(G_\alpha) \circ \varphi_\alpha = g \circ \varphi_\alpha^{-1} \circ \varphi_\alpha = g.$$

Therefore, the function $F_\alpha : \Delta_\alpha \to \mathbb{C}$ defined by $F_\alpha = e^{-H_\alpha}$ is such that

$$|F_\alpha(p)| = |e^{-\text{Re}(H_\alpha(p))} e^{-i\text{Im}(H_\alpha(p))}| = e^{-g(p,q)}.$$

Note that $\text{Im}(H_\alpha)$ is unique up to an additive real constant, hence F_α is unique up to multiplication by $e^{i\theta}$ for some $\theta \in \mathbb{R}$.

Now select the a chart $(\Delta_\alpha, \varphi_\alpha)$ such that $q \in \Delta_\alpha$. Then the function

$$f(p) = g(p,q) + \log |\varphi_\alpha(p) - \varphi_\alpha(q)|$$

is real harmonic in Δ_α. By the same procedure as above, there exists an analytic function $H_\alpha : \Delta_\alpha \to \mathbb{C}$ such that $\text{Re}(H_\alpha(p)) = f(p)$. Define the function $F_\alpha : \Delta_\alpha \to \mathbb{C}$ as

$$F_\alpha(p) = (\varphi_\alpha(p) - \varphi_\alpha(q)) e^{-H_\alpha(p)}.$$

Then we have

$$|F_\alpha(p)| = |\varphi_\alpha(p) - \varphi_\alpha(q)| e^{-\text{Re}(H_\alpha(p))} = |\varphi_\alpha(p) - \varphi_\alpha(q)| e^{-g(p,q) - \log |\varphi_\alpha(p) - \varphi_\alpha(q)|} = e^{-g(p,q)}.$$

Therefore, we have a collection of functions $\{F_\alpha\}$ that are unique up to multiplication by $e^{i\theta}$ for some $\theta \in \mathbb{R}$ such that $|F_\alpha(p)| = e^{-g(p,q)}$. It follows that if $\Delta_\alpha \cap \Delta_\beta \neq \emptyset$, then $|F_\alpha| = |F_\beta|$ on $\Delta_\alpha \cap \Delta_\beta$. Hence

$$\frac{F_\alpha}{F_\beta} = e^{i\theta},$$

for some $\theta \in \mathbb{R}$ on $\Delta_\alpha \cap \Delta_\beta$.

Therefore, $F_\alpha = e^{i\theta} F_\beta$ is an analytic continuation of F_α from $\Delta_\alpha \cap \Delta_\beta$ to Δ_β. It follows that F_α can be analytically extended along any path in R starting in U_α. By the Monodromy theorem, F_α admits an analytic continuation to R, and by construction, $|F_\alpha(p)| = e^{-g(p,q)}$.

□

Note that the F given above only has one zero (at q), and $|F(p)| < 1$ for all $p \in R$ since $g(p,q) > 0$.

Theorem 4.1. *Let R be a simply connected Riemann surface such that a Green's function $g(p,q)$ exists. Then there exists a conformal map from R to the open unit disc.*

Proof. Fix $q_0 \in R$, and obtain a Green's function $g(p, q_0)$ with pole at q_0. Let $F : R \to \mathbb{C}$ be an analytic function such that $|F(p)| = e^{-g(p,q_0)}$. Fix $q_1 \in R$ such that $q_1 \neq q_0$. Define $\varphi : R \to \mathbb{C}$

$$\varphi(p) = \frac{F(p) - F(q_1)}{1 - \overline{F(q_1)}F(p)}.$$

Since $|F(p)| < 1$ for all $p \in R$, φ is well-defined. Furthermore, φ is analytic, $\varphi(q_1) = 0$, and $\varphi(q_0) = -F(q_1)$. Next, we show that $|\varphi(p)| < 1$ for any given $p \in R$. Let $|F(p)|^2 = 1 - \varepsilon_0$ and $|F(q_1)|^2 = 1 - \varepsilon_1$, where $\varepsilon_i \in (0, 1]$. Then we have

$$|F(p)|^2 + |F(q_1)|^2 = (1 - \varepsilon_0) + (1 - \varepsilon_1) < (1 - \varepsilon_0) + (1 - \varepsilon_1) + \varepsilon_0\varepsilon_1 = 1 + |F(p)|^2|F(q_1)|^2.$$

From this identity, we can derive

$$\begin{aligned} |F(p) - F(q_1)|^2 &= |F(p)|^2 - F(p)\overline{F(q_1)} - \overline{F(p)}F(q_1) + |F(q_1)|^2 \\ &< 1 - F(p)\overline{F(q_1)} - \overline{F(p)}F(q_1) + |F(p)|^2|F(q_1)|^2 \\ &= |1 - F(p)\overline{F(q_1)}|^2. \end{aligned}$$

Hence we have shown $|\varphi(p)| < 1$.

Next, we show that φ is injective. If $u \in \mathcal{P}_{q_1}$, then $u(p) + \log|\varphi(p)|$ is subharmonic, and furthermore $u(p) + \log|\varphi(p)| < 0$ on the complement of the compact support of u. By the maximum principle, $u(p) + \log|\varphi(p)| < 0$ everywhere in R.

Since $g(p, q_1)$ is the supremum over all such functions $u \in \mathcal{P}_{q_1}$, we obtain

$$g(p, q_1) + \log|\varphi(p)| \leq 0$$

for all $p \in R$.

In particular, we have $g(q_0, q_1) + \log|\varphi(q_0)| \leq 0$. We can also write

$$g(q_0, q_1) + \log|\varphi(q_0)| = g(q_0, q_1) + \log|F(q_1)| = g(q_0, q_1) - g(q_1, q_0).$$

Therefore, $g(q_0, q_1) - g(q_1, q_0) \leq 0$. However, we can repeat this entire argument while writing q_0 in place of q_1 and vice-versa, and obtain $g(q_1, q_0) - g(q_0, q_1) \leq 0$. Hence we have shown symmetry of Green's function, and in particular,

$$g(q_0, q_1) + \log|\varphi(q_0)| = g(q_0, q_1) - g(q_1, q_0) = 0.$$

But earlier we showed that the subharmonic function $g(p, q_1) + \log|\varphi(p)|$ is less than or equal to zero on all of R, hence by the maximum principle it is identically zero and

$$g(p, q_1) = -\log|\varphi(p)|.$$

Since $g(p, q_1)$ is finite on $R\backslash\{q_1\}$, we conclude that φ has no zeroes in $R\backslash\{q_1\}$. Therefore, $F(p) - F(q_1) \neq 0$ in $R\backslash\{q_1\}$, and hence $F(p) = F(q_1)$ if and only if $p = q_1$. Since q_1 was chosen arbitrarily, we conclude that F is one-to-one.

It follows that $F : R \to F(R) \subseteq \mathbb{D}$ is conformal, and since $F(R)$ is a simply connected open set, there is a conformal map from $F(R)$ to the open disc by the Riemann mapping theorem. Therefore there is a conformal map from R to the open disc.

□

Corollary 4.1. *Let R be a Riemann surface such that a Green's function exists. Then for all distinct $p, q \in R$, $g(p, q) = g(q, p)$.*

Proof. This was shown while proving the previous theorem. □

5. Bipolar Green's Functions

In this section, we classify simply connected Riemann surfaces R that do not admit a Green's function. It will be shown that in this case, we can construct a conformal map to the Riemann sphere if R is compact, or a conformal map to the complex plane otherwise.

Definition 5.1. *Let q_1, q_2 be distinct points in a Riemann surface R. Let (Δ_1, φ_1) and (Δ_2, φ_2) be coordinate discs such that $\Delta_1 \cap \Delta_2 = \emptyset$, and $\varphi_1(q_1) = \varphi_2(q_2) = 0$. A harmonic function $G(p, q_1, q_2)$ on $R \backslash \{q_1, q_2\}$ such that*

* $G(p, q_1, q_2) + \log|\varphi_1(p)|$ *is harmonic at* q_1,
* $G(p, q_1, q_2) - \log|\varphi_2(p)|$ *is harmonic at* q_2, *and*
* $G(p, q_1, q_2)$ *is bounded on* $R \backslash (\Delta_1 \cup \Delta_2)$,

is called a **bipolar Green's function** *for R with poles at q_1 and q_2.*

Proposition 5.1. *For all points $q_1, q_2 \in R$ such that $q_1 \neq q_2$, there exists a bipolar Green's function $G(p, q_1, q_2)$.*

Proof. Let S be a Riemann surface with an atlas that consists of finitely many precompact coordinate discs, whose closure is contained in a Riemann surface R. Further assume that R is an open set of a Riemann surface R' and that R has an atlas that consists of finitely many precompact coordinate discs. We will start by deriving results with these assumptions, then move to the general case by taking countable unions of coordinate discs.

Let q_1, q_2 be distinct points in S, and let (U_1, φ_1) and (U_2, φ_2) be charts such that $q_1 \in U_1$, $\varphi_1(q_1) = 0$, $q_2 \in U_2$, $\varphi_2(q_2) = 0$, and $U_1 \cap U_2 = \emptyset$. Choose $r > 0$ such that $\Delta_i = \{p \in U_i : |\varphi_i(p)| \leq r\}$ is well defined. Select $s > 0$ such that $r > s$ and $\sigma_i = \{p \in U_i : |\varphi_i(p)| \leq s\}$ is well defined. We know Green's functions exist on R by Proposition 4.4. The first objective is to give a bound on the difference of the Green's functions $g_R(p, q_1)$ and $g_R(p, q_2)$ for $p \in R \backslash (\Delta_1 \cup \Delta_2)$, and furthermore we want this bound to be independent of R.

Let $M_i = \max_{p \in \partial\sigma_i} g_R(p, q_i)$. Since $\inf_{p \in R} g(p, q_i) = 0$, we have $g(p, q_i) \to 0$ as $p \to \partial R$. Now $g(p, q_i)$ is a harmonic function on $R \backslash (\sigma_1 \cup \sigma_2)$, so we can apply the maximum principle to conclude $g_R(p, q_i) \leq M_i$ for $p \in R \backslash (\sigma_1 \cup \sigma_2)$. In particular, $M_i - g_R(p, q_i) \geq 0$ for $p \in S \backslash (\sigma_1 \cup \sigma_2)$.

Let $z_i \in \partial\Delta_i$ be such that $g(p, q_i) \leq g(z_i, q_i)$ for all $p \in \partial\Delta_i$. This is possible by compactness of $\partial\Delta_i$. Since $g_R(p, q_i) + \log|\varphi_i(p)|$ is harmonic on Δ_i, we apply the maximum principle to conclude

$$g_R(p,q_i) + \log|\varphi_i(p)| \leq g(z_i,q_i) + \log r$$

for all $p \in \Delta_i$. In particular, we can take the maximum over all values in $\partial\sigma_i$ and obtain:

$$M_i - g(z_i,q_i) \leq \log r/s.$$

Since $M_i - g_R(p,q_i)$ is a non-negative harmonic function on $S\backslash(\sigma_1 \cup \sigma_2)$ and $\partial\Delta_1 \cup \partial\Delta_2$ is compact, we can apply Harnack's estimate to obtain a $C > 0$ such that

$$\frac{M_i - g_R(p,q_i)}{M_i - g_R(z_i,q_i)} \leq C$$

for all $p \in \partial\Delta_1 \cup \partial\Delta_2$. Therefore,

$$M_i - g_R(p,q_i) \leq C(M_i - g_R(z_i,q_i)) \leq C\log r/s = C_0,$$

for $p \in \partial\Delta_1 \cup \partial\Delta_2$, where C_0 is independent of R. We can rewrite this as

$$M_i - C_0 \leq g_R(p,q_i) \leq M_i$$

for all $p \in \partial\Delta_1 \cup \partial\Delta_2$. Therefore,

$$|g_R(p,q_1) - g_R(p,q_2)| \leq M_1 - (M_2 - C_0) = (M_1 - M_2) + C_0$$

for all $p \in \partial\Delta_1 \cup \partial\Delta_2$. We know that $\inf_{p\in R} g(p,q_i) = 0$, so $g(p,q_i) \to 0$ as $p \to \partial R$, hence applying the maximum yields the estimate for all $p \in R\backslash(\Delta_1 \cup \Delta_2)$. However, the M_i depend on R. We must work a little harder in order to obtain a bound independent of R.

Since $g_R(p,q_1)$ is harmonic for $p \in \Delta_2$, we can apply the maximum principle on the boundary of Δ_2 to conclude $M_1 - C_0 \leq g_R(p,q_1) \leq M_1$ for all $p \in \Delta_2$. In particular,

$$M_1 - C_0 \leq g_R(q_2,q_1) \leq M_1.$$

The same argument also yields

$$M_2 - C_0 \leq g_R(q_1,q_2) \leq M_2.$$

Since $g_R(q_1,q_2) = g_R(q_2,q_1)$, we have

$$M_1 - C_0 \leq g_R(q_1,q_2) \leq M_2.$$

Therefore $|M_1 - M_2| \leq C_0$, which is independent of R. Hence

$$|g_R(p, q_1) - g_R(p, q_2)| \leq 2C_0$$

for all $p \in R \backslash \Delta_1 \cup \Delta_2$.

We are now equipped to deal with an arbitrary Riemann surface M. Let q_1, q_2 be distinct points in M, and (Δ_1, φ_1), (Δ_2, φ_2) coordinate discs such that $\Delta_1 \cap \Delta_2 = \emptyset$, $q_1 \in \Delta_1$, $\varphi_1(q_1) = 0$, $q_2 \in \Delta_2$, and $\varphi_2(q_2) = 0$. By Proposition 3.1, we have

$$M = \bigcup_{i=1}^{\infty} S_i,$$

where S_i is a Riemann surface who has an atlas consisting of finitely many precompact coordinate discs. By taking finite unions are relabelling, we can assume $\Delta_1 \cup \Delta_2 \subset S_1$, and $\overline{S_i} \subseteq S_{i+1}$. For each S_n, obtain Green's functions $g_i(p, q_1)$ and $g_i(p, q_2)$ and create the sequence

$$G_n(p, q_1, q_2) = g_n(p, q_1) - g_n(p, q_2).$$

As shown above, $|G_n(p, q_1, q_2)| \leq 2C_0$ for all n, hence we have a sequence of uniformly bounded harmonic functions. By Proposition 2.7, there is a subsequence $\{G_{n_k}\}$ that converges to a harmonic function $G(p, q_1, q_2)$ on $M \backslash (\Delta_1 \cup \Delta_2)$.

Also, $g_{n_k}(p, q_1) + \log|\varphi_1(p)| - g_{n_k}(p, q_2) = G_{n_k}(p, q_1, q_2) + \log|\varphi_1(p)|$ is harmonic in Δ_1, and on $\partial\Delta_1$ we have the uniform bound

$$|G_{n_k}(p, q_1, q_2) + \log|\varphi_1(p)|| = |g_{n_k}(p, q_1) - g_{n_k}(p, q_2)| \leq 2C_0.$$

By the maximum principle, the bound holds in Δ_1. Therefore, we have convergence to a harmonic function $G(p, q_1, q_2) + \log|\varphi(p)|$ on Δ_1. The same process can be repeated on Δ_2. Hence $G(p, q_1, q_2)$ is the bipolar Green's function for M.

□

Lemma 5.1. *Let R be a Riemann surface, and suppose there exists a nonconstant, bounded, complex-valued analytic function on R. Then there exists a Green's function on R.*

Proof. Fix $q \in R$. Let h be a nonconstant complex-valued analytic function such that $|h| < B_0$. Then

$$\varphi(p) = \frac{h(p) - h(q)}{B_0}$$

is an analytic function such that $\varphi(q) = 0$ and $|\varphi| < 1$. For any $u \in \mathcal{P}_q$, then $u(p) + \log|\varphi(p)|$ is a subharmonic function such that $u(p) + \log|\varphi(p)| < 0$ for all p outside the compact support of u. By the maximum principle, we have

$$u(p) < -\log|\varphi(p)|.$$

Since this holds for all $u \in \mathcal{P}_q$, the supremum is not identically infinite and hence a Green's function exists.

□

Theorem 5.1. *Let R be a simply connected Riemann surface such that a Green's function $g(p, q)$ does not exists. Then either there exists a conformal map from R to the complex plane, or there exists a conformal map from R to the Riemann sphere.*

Proof. Fix q_1, q_2 distinct points in R. Let $G(p, q_1, q_2)$ be a bipolar Green's function for R. By making slight modifications to the proof of Lemma 4.1, we can see that there exists a meromorphic function $F : R \to \mathbb{C} \cup \{\infty\}$ such that

$$|F(p)| = e^{-G(p,q_1,q_2)}.$$

Notice that F has one simple zero at q_1 and one simple pole at q_2. We want to show that F is injective. Let $q_0 \in R$ be distinct from q_1, q_2. Let $H : R \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function such that

$$|H(p)| = e^{-G(p,q_0,q_2)}.$$

From the definition of bipolar Green's function, F is bounded away from q_1, q_2 and H is bounded away from q_0, q_2. Also, we have $F(q_1) = H(q_2) = 0$ and $F(q_2) = H(q_2) = \infty$. Since all poles and zeros are simple, the function $\varphi : R \to \mathbb{C}$ defined as

$$\varphi(p) = \frac{F(p) - F(q_0)}{H(p)}$$

is well-defined, as well as analytic and bounded. By the lemma, $\varphi \equiv$ *const*. Since $\varphi(q_1) = -F(q_0)/H(q_1) \neq 0$, we see that φ is nonzero. Hence $F(p) = F(q_0)$ if and only if $p = q_0$. Since q_0 was arbitrary, we have proved that F is injective.

Denote the image of F as $\Omega \subseteq \mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$. We know that Ω is open and simply connected, and $F : R \to \Omega$ is conformal. If $\hat{\mathbb{C}} \backslash \Omega$ has more than one point, we can move a point in the complement of Ω to ∞ via a fractional transformation, and obtain a conformal map from Ω to a subset of $\mathbb{C}$ that is not all of the complex plane. By the Riemann mapping theorem, Ω can be mapped conformally to the open unit disc, and hence R can be conformally mapped to the open unit disc. But then there exists a non-constant, bounded analytic function on R; by the lemma, a Green's function must exist. This is a contradiction, and hence $\hat{\mathbb{C}}\backslash\Omega$ is either a single point, or the empty set.

If $\hat{\mathbb{C}}\backslash\Omega$ has exactly one point, then we can move that point to ∞ via a fractional transformation and hence obtain a conformal map from Ω to all of $\mathbb{C}$. If Ω is all of $\hat{\mathbb{C}}$, then F maps R conformally to the Riemann sphere.

□

References

[1] W. Abikoff. The uniformization theorem. *The American Mathematical Monthly*, 88(8):574–592, 1981.

[2] T. Carne. Riemann surfaces and discrete groups: Course lecture notes. url: http://www.dpmms.cam.ac.uk/ tkc/, Lent 2007.

[3] T. W. Gamelin. *Complex Analysis*, volume 1. Springer-Verlag, New York, 1st edition, 2001.

[4] J. Lee. *Introduction to Smooth Manifolds*, volume 1. Springer, New York, 2nd edition, 2006.