MATH 580 TAKE-HOME MIDTERM EXAM

DUE THURSDAY NOVEMBER 3

1. Consider the initial value problem
$$u_t + uu_x = u^2, \qquad u(x,0) = g(x).$$
Prove that a solution u satisfies
$$u(x,t) = \frac{g(\xi)}{1 - tg(\xi)}, \qquad \text{with} \qquad x = \xi - \log(1 - tg(\xi)).$$
Prove that if $g \in C^1(\mathbb{R})$ and $\|g\|_{\infty,\mathbb{R}} + \|g'\|_{\infty,\mathbb{R}} < \infty$, then there exists $T > 0$ such that the initial value problem has a unique C^1 solution defined on $\mathbb{R} \times (-T, T)$. Show that if g is given by
$$g(x) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 \le x \le 1, \\ 0, & x \ge 1, \end{cases}$$
then not only we have the issues caused by the multi-valuedness, but also that $u(x,t) \to \infty$ for $x < 0$ as $t \to 1$.
2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary, and let $k \in C(\partial\Omega)$ with $k \ge 0$. Show that the following problem has at most one solution $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$.
$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \\ \partial_\nu u + k\Delta u = h & \text{on } \partial\Omega. \end{cases}$$
Explicitly solve the problem for
$$\begin{aligned} \Omega = \{x \in \mathbb{R}^2 : 1 < |x| < e\}, \quad k \equiv 0, \quad f \equiv 0, \\ g = 0 \quad \text{and} \quad h = -1 \quad \text{on } \{|x| = 1\}, \\ g = e^2 \quad \text{and} \quad h = 3e \quad \text{on } \{|x| = e\}. \end{aligned}$$
3. Prove that a function subharmonic in $\mathbb{R}^2$ and bounded from above must be constant. Is this statement true in $\mathbb{R}^n$ for $n \ge 3$?
4. Let Ω be a domain, and let $\Sigma = \partial\Omega \cap B$ be a smooth and nonempty portion of the boundary, where B is an open ball. Let $u \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma)$ satisfy $\Delta u = 0$ in Ω and $u = \partial_\nu u = 0$ on Σ. Show that u is identically zero in Ω.

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5. Let $\Omega \subset \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ be a domain, and let $\Sigma := \{x \in \partial\Omega : x_n = 0\}$ be a nonempty open subset of the hyperplane $\{x_n = 0\}$. Suppose that $u \in C^2(\Omega) \cap C(\Omega \cup \Sigma)$ is harmonic in Ω and $u = 0$ on Σ. Denote by x^* the reflection $(x_1, \ldots, x_{n-1}, -x_n)$ of $x = (x_1, \ldots, x_{n-1}, x_n)$, and let $\tilde{\Omega} = \Omega \cup \Sigma \cup \{x^* : x \in \Omega\}$. Then prove that the function $\tilde{u} \in C(\tilde{\Omega})$ defined by $\tilde{u} = u$ in $\Omega \cup \Sigma$ and $\tilde{u}(x^*) = -u(x)$ for $x \in \Omega$ is harmonic in $\tilde{\Omega}$. Use this to show that
   (a) The Cauchy problem for the Laplace equation $\Delta u = 0$ with the Cauchy data $u = 0$ and $\partial_n u = g$ on the hyperplane $\{x_n = 0\}$ has no solution in any neighbourhood of $0 \in \mathbb{R}^n$, if g is not analytic at $0 \in \mathbb{R}^{n-1}$.
   (b) A bounded harmonic function in $\mathbb{R}^n_+$ with $u = 0$ on $\{x_n = 0\}$ is identically zero.
6. Let $\Omega \subset \mathbb{R}^n$ be bounded, and let $\alpha \in (0, 1]$. We denote by $C^{0,\alpha}(\Omega)$ the *Hölder space*, that is, the space of functions $u \in C(\Omega)$ for which the *Hölder norm*

$$\|u\|_{C^{0,\alpha}(\Omega)} := \sup_{x \in \Omega} |u(x)| + \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

   is finite.
   (a) Show that $C^{0,\alpha}(\Omega)$ is a Banach space, i.e., that it is a linear space, $\|\cdot\|_{C^{0,\alpha}(\Omega)}$ is a norm on it, and it is complete with respect to $\|\cdot\|_{C^{0,\alpha}(\Omega)}$.
   (b) Show that the unit ball in $C^{0,\alpha}(\Omega)$ is compact in $C_b(\Omega)$, the latter being the space of bounded continuous functions on Ω with the uniform norm.