Existence of PDEs Through Finite Difference Methods
MATH 580 - Final Project

Olivier Mercier

December 14, 2011

We show existence of solutions for second order linear equations of hyperbolic, parabolic and elliptic type. We use the finite difference approach, which uses a discrete approximation of the PDEs on a grid. The general goal is to show that as the grid size tends to zero, the solution of the discretized equation can be used to construct a continuous solutions of the PDEs.

1 Linear advection

We begin with a trivial case to set the notation and show the basic principles that will be used for more complicated cases.

1.1 Exact solution

We want to solve the equation

$$u_t + u_x = 0 \quad \text{on}\, \mathbb{R}^2 \tag{1}$$

$$u(x,0) = f(x) \tag{2}$$

Using the method of characteristics, the solution is easily found to be

$$u(x,t) = f(x-t). \tag{3}$$

This means that the solution at a point (x,t) is completely determined by the information at the point $(x-t,0)$. We thus expect any scheme approximating this PDE to have a similar behavior.

1.2 Naive discretization

We want to discretize the problem (1)-(2) to have an approximated solution v on a grid Σ. For simplicity, we only care about positive time t. We define the grid, for fixed values h and k, to be

$$\Sigma := \{(ih, jk) \,|\, i \in \mathbb{Z}, j \in \mathbb{N}\}$$

The simplest way to discretize the PDE (1) is to replace the derivatives using forward Euler differences. This gives the scheme

$$\frac{v(x,t+k)-v(x,t)}{k}+\frac{v(x+h,t)-v(x,t)}{h}=0 \quad \text{for } (x,t)\in\Sigma \tag{4}$$

$$v(x,0)=f(x) \quad \text{for } (x,0)\in\Sigma. \tag{5}$$

This scheme formally approximates the PDE, in the sense that as $h,k\to 0$, (4)-(5) becomes (1)-(2). But it is unclear that a solution of the discrete scheme will give a solution of the continuous PDE. It isn't even clear how to create a function u defined on $\mathbb{R}^2$ from v, especially for irrational points.

We can still analyse the discrete scheme and keep those convergence questions for later. Since the transport equation at a time $(t+k)$ is simply the transported information at the time t, we would like to write our scheme in a more general way in terms of $v(x,t+k)$ and $v(x,t)$ only. To do this we define the *spatial shift operator*

$$Eg(x)=E_hg(x):=g(x+h).$$

We can then write (4) as

$$\begin{aligned} v(x,t+k) &= \left(1+\frac{k}{h}\right)v(x,t)-\frac{k}{h}v(x+h,t) \\ &= \left(1+\frac{k}{h}-\frac{k}{h}E\right)v(x,t). \end{aligned} \tag{6}$$

Doing this allows us to have an explicit expression for $v(x,nk)$, ensuring that the discrete scheme can be solved for any point of Σ. But we still need to check if v converges to a solution of the PDE. Iterating (6) and using the binomial expansion, we get

$$\begin{aligned} v(x,mk) &= \left(1+\frac{k}{h}-\frac{k}{h}E\right)^m v(x,0) \\ &= \left(1+\frac{k}{h}-\frac{k}{h}E\right)^m f(x) \\ &= \sum_{i=0}^{m}\binom{m}{i}\left(1+\frac{k}{h}\right)^i\left(-\frac{k}{h}\right)^{m-i}E^{m-i}f(x) \\ &= \sum_{i=0}^{m}\binom{m}{i}\left(1+\frac{k}{h}\right)^i\left(-\frac{k}{h}\right)^{m-i}f(x+(m-i)h) \end{aligned} \tag{7}$$

Written in this form, it is clear that the solution $v(x,mk)$ is constructed using only the information at the points $\{(x+ih,t+jk) : 0\le i\le m-j\}$, and ultimately using only the information of $f(x+ih)$ for $0\le i\le m$, as shown in figure 1. But since the exact solution $u(x,t)$ for the PDE depends only on the information of $f(x-t)$, there is no way v could converge to the solution u. This argument can be made rigorous by looking at the Courant-Friedrichs-Lewy (CFL) condition for the domain of dependence of a scheme.

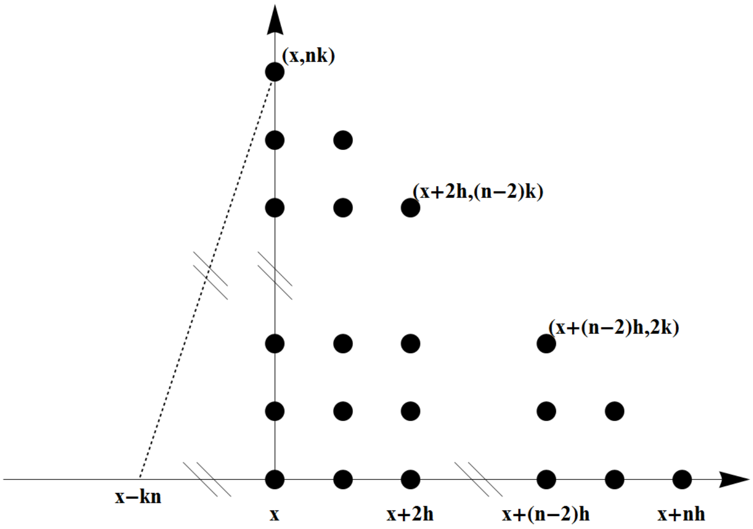

Figure 1: Dependance domain of the solutions. The characteristic for the PDE at the point (x, mk) is shown as a dashed line. All the black dots are the points that affect the discrete solution at (x, mk). We see that those point are not related to the information on the characteristic line for the PDE.

1.3 Stable discretization

To solve this problem, we need to use a more complicated difference scheme to discretize the equation. Instead of (4)-(5), we use a backward Euler difference for the space derivative, which yield the scheme

$$\frac{v(x,t+k)-v(x,t)}{k} + \frac{v(x,t)-v(x-h,t)}{h} = 0 \quad \text{for } (x,t) \in \Sigma \tag{8}$$

$$v(x,0) = f(x) \quad \text{for } (x,0) \in \Sigma. \tag{9}$$

and the solution can be written explicitly as

$$v(x,mk) = \sum_{i=0}^{m} \binom{m}{i} \left(1-\frac{k}{h}\right)^{i} \left(\frac{k}{h}\right)^{m-i} f(x-(m-i)h) \tag{10}$$

$$\tag{11}$$

and now, we see that the domain of dependence of the solution at (x, mk) will include the characteristic line if $\frac{k}{h} \leq 1$. Then, by sending the grid size to 0, there is a chance the discrete solution will converge to the exact solution. Since this ratio $\frac{k}{h}$ is important to the correctness of the scheme, we define $\lambda := \frac{k}{h}$. It is called the *Courant number* for this difference scheme, and the inequality $\lambda \leq 1$ is called the *CFL condition* for this scheme.

This λ plays an important role in the stability of the scheme, and the Lax-Richtmyer theorem could be employed to show convergence of the scheme, but we won't go into any details here since the goal is to introduce the techniques for more complicated cases. In those cases we won't know *a priori* the existence of a solution to the PDE, so we will need to proceed differently.

To show convergence, since here we know explicitly the exact solution, we can directly compare it with the discrete solution. If the CFL condition holds, and supposing $||f''|| < \infty$, we have

$$\begin{aligned} & |u(x, t+k) - v(x, t+k)| \\ = & |f(x-t-k) - (1-\lambda)f(x-t) - \lambda f(x-t-h)|. \end{aligned} \tag{12}$$

Developing f around $(x-t)$, we get, for some ξ_1 and ξ_2,

$$\begin{aligned} = & \left| f(x-t) - kf'(x-t) + \frac{k^2}{2}f''(\xi_1) - (1-\lambda)f(x-t) - \lambda(f(x-t) - hf'(x-t) + \frac{h^2}{2}f''(\xi_2)) \right| \\ = & \left| \frac{k^2}{2}f''(\xi_1) + \lambda\frac{h^2}{2}f''(\xi_2) \right| \\ = & \left| \frac{\lambda^2 h^2}{2}f''(\xi_1) + \lambda\frac{h^2}{2}f''(\xi_2) \right| \\ \leq & h^2 \frac{\lambda^2 + \lambda}{2} \sup |f''| \end{aligned} \tag{13}$$

Now, define $w := u - v$. By taking $m = 1$ in (10) we get

$$v(x, t+k) - (1-\lambda)v(x,t) - \lambda v(x-h, t) = 0$$

and so we have by (13) the estimate

$$|w(x, t+k) - (1-\lambda)w(x,t) - \lambda w(x-h,t)| \leq h^2 \frac{\lambda^2 + \lambda}{2} \sup |f''|$$

which yields

$$\begin{aligned} \sup_x |w(x, t+k)| \leq & (1-\lambda)\sup_x |w(x,t)| + \lambda \sup_x |w(x-h,t)| + h^2 \frac{\lambda^2 + \lambda}{2} \sup |f''| \\ = & \sup_x |w(x,t)| + h^2 \frac{\lambda^2 + \lambda}{2} \sup |f''|. \end{aligned}$$

Recursively applying this estimate gives

$$\sup_x |w(x, mk)| \leq \sup_x |w(x,0)| + mh^2 \frac{\lambda^2 + \lambda}{2} \sup |f''|$$

and since w(x,0) = 0,

$$= mh^2 \frac{\lambda^2 + \lambda}{2} \sup |f''| = \frac{t}{k} h^2 \frac{\lambda^2 + \lambda}{2} \sup |f''| = \frac{th}{\lambda} \frac{\lambda^2 + \lambda}{2} \sup |f''|$$

and so we have that for fixed λ and t, $w(x,t) \to 0$ as $h, k \to 0$. This shows that the solution to the difference equation tends to the exact solution of the PDE as the grid size goes to 0.

For this trivial case, we already knew the existence and uniqueness of the solution, and so showing that the solution of the discrete scheme tends to this exact solution was relatively easy. For the next case, we will not know *a priori* the existence of a solution to the PDE, and we will need to construct this solution from the solution of the discrete scheme, which will require much more work.

2 Linear first order symmetric systems of hyperbolic equations

We want to use here the same ideas used in section 1 for the case of a linear first order system of hyperbolic equations. Namely, we want to solve the system

$$A_0(\vec{x}, t)\partial_t u + \sum_{j=1}^{n} A_j(\vec{x}, t)\partial_j u + B(\vec{x}, t)u = w(\vec{x}, t) \tag{14}$$

along with given initial data $u(\vec{x}, 0)$ for a function $u = (u_1(\vec{x}, t), ..., u_M(\vec{x}, t)) : \Omega = \mathbb{R}^n \times (0, T) \to \mathbb{R}^M$. A symmetric hyperbolic system means here that all the A_j are symmetric and A_0 is positive definite. The interest of solving such a system is mainly because a lot of higher order hyperbolic equations can be transformed in such a system. In particular, any second order hyperbolic PDE can be written as such a system.

2.1 Notation

We extend the notation used in section 1 in higher dimensions. We will work on the grid Σ defined for fixed lengths h and k by

$$\Sigma := \{(\alpha_1 h, ..., \alpha_n h, jk) \,|\, \alpha_i \in \mathbb{Z} \; \forall i, \; j \in \mathbb{N}\}.$$

We will use a multi index notation defined by

$$(x_1, ..., x_n, t) = (\alpha_1 h, ..., \alpha_n h, jk) =: (\alpha h, jk).$$

As we did previously, we will need the shift operators

$$E_0 g(x_1, ..., x_n, t) = g(x_1, ..., x_n, t + k)$$
$$E_j g(x_1, .., x_n, t) = g(x_1, ..., x_j + h, ..., x_n, t)$$

with their inverses

$$E_0^{-1} g(x_1, ..., x_n, t) = g(x_1, ..., x_n, t-k)$$
$$E_j^{-1} g(x_1, .., x_n, t) = g(x_1, ..., x_j - h, ..., x_n, t)$$

and for clarity, we also define the divided difference operators

$$\delta_0 = \frac{E_0 - 1}{k}$$
$$\delta_j = \frac{E_j - 1}{h}.$$

We will often omit the arguments of the function and write $E_0 g(\vec{x}, t) =: E_0 g$. We will also apply the multi index to these operators to get

$$\delta_0^i \delta^\alpha g := \delta_0^i \delta_1^{\alpha_1} ... \delta_n^{\alpha_n} g$$

2.2 Existence proof

2.2.1 Behavior of the discrete scheme

The first thing to do is to find a suitable discretization of the system (14). As was exposed in section 1, the most intuitive schemes sometime don't have the properties required to converge to the solution of the PDE. For instance, in this case, one would be tempted to discretize (14) by

$$A_0 \delta_0 v + \sum_{j=1}^{n} A_j \delta_j v + Bv = w \tag{15}$$

but for reasons similar to those exposed by figure 1, this scheme will not be stable and won't give us a correct solution to the PDE. We will instead use a scheme which replaces the space derivatives by the central difference quotients

$$\frac{E_j - E_j^{-1}}{2h}$$

The simplest scheme using these central differences would be

$$\frac{1}{k} A_0 (E_0 - 1) v + \frac{1}{2h} \sum_{j=1}^{n} A_j (E_j - E_j^{-1}) v + Bv = w$$

but it turns out that this isn't stable enough yet. One has to replace the values of v in the leftmost term by an average along the neighboring points in space. This can be interpreted as a way of increasing the continuity of the solution to the scheme. Taking this average on the two closest points in every variable, we get the scheme

$$\frac{1}{k} A_0 \left(E_0 - \frac{1}{2n} \sum_{j=1}^{n} (E_j + E_j^{-1}) \right) v + \frac{1}{2h} \sum_{j=1}^{n} A_j (E_j - E_j^{-1}) v + Bv = w \tag{16}$$

and this scheme is finally correct and it is the one we will use. Since the matrix A_0 is positive definite, it is invertible, and we can rewrite (16) as

$$v(\vec{x}, t+k) = kA_0^{-1}\left(\left(\sum_{j=1}^{n}\left[\frac{1}{2n}A_0(E_j + E_j^{-1}) - \frac{1}{2h}A_j(E_j - E_j^{-1})\right] - B\right)v + w\right). \tag{17}$$

This gives us an explicit expression for $v(\vec{x}, t+mk)$ for any $m \in \mathbb{N}$ by iterating (17) m times, and thus proves the existence of a solution v to the finite difference scheme. We now want to ensure that this discrete solution behaves well when we sent the grid size to zero.

Define $\lambda := \dfrac{k}{h}$ to be the Courant number for this scheme. We won't go into the details here, but it is possible, by defining the right energies, to get the estimates

$$h^n k \sum_{(x,t)\in\Sigma} (v(x,t))^T A(x,t) v(x,t) \leq \gamma T e^{CT} \sum_{(x,t)\in\Sigma} (w(x,t))^T A(x,t) w(x,t) \tag{18}$$

$$\max_{(x,t)\in\Sigma} |\delta^\alpha v(x,t)|^2 = O\left(\sum_{\beta|\leq|\alpha|+n} \int_\Omega \left|\partial^\beta w(x,t)\right|^2 d\vec{x}dt\right) \tag{19}$$

for small enough values of λ and some constants γ and C. See [1] for details on those calculations. If w is smooth enough, which we will assume, then these estimates give us a bound on v and its derivatives that doesn't depend on the grid size. Using the Sobolev embedding theorem, we can go from an L_2 bound to an L_∞ bound to have

$$\max_{(x,t)\in\Sigma} |\partial^\alpha v(x,t)| < \infty \quad \forall\, 0 \leq |\alpha| \leq 2. \tag{20}$$

which means that the $|\partial^\alpha v(x,t)|$ aren't only bounded, they are Lipschitz $\forall\,|\alpha| \leq 1$.

2.2.2 Refining the grid

Now that we know the discrete function v has a good behavior, we are ready to shrink the grid size. Fix λ small enough do that (20) holds and take $h = 2^{-q}$ and $k = \lambda 2^{-q}$ for $q \in \mathbb{N}$. Doing so gives us a sequence of grids Σ_q such that

$$\Sigma_p \subset \Sigma_q \quad \text{if } p < q.$$

Define v_q to be the solution on Σ_q and $\sigma = \cup_q \Sigma_q$. Since v and its first order derivatives are bounded, there is a subset $S_1 \subset \mathbb{N}$ such that the limit

$$\lim_{\substack{q\in S_1\\ q\to\infty}} v_q(x,t)$$

exists for $(x,t) \in \sigma$. We can then take a subset $S_2 \subset S_1$ so that the limit

$$\lim_{\substack{q \in S_2 \\ q \to \infty}} \delta_0 v_q(x,t)$$

exists, and so on, and we can take the last of these subsets, call it S, so that all the limits converge for $q \in S, q \to \infty$, and define

$$\lim_{\substack{q \in S \\ q \to \infty}} \delta_0^i \delta_x^\alpha v_q(x,t) =: u^{i,\alpha}(x,t) \quad \forall\, |\alpha| + i \leq 1$$

for $(x,t) \in \sigma$. Since the set σ is dense in Ω, every point not in σ is the limit of a sequence of points $(x_q, t_q) \in \Sigma_q$ for $S \ni q \to \infty$. And since the functions $u^{i,\alpha}$ are Lipschitz on σ, we can extend the functions $u^{i,\alpha}$ to points $(x,t) \notin \sigma$ to a Lipschitz function on Ω by

$$u^{i,\alpha}(x,t) = \lim_{\substack{q \in S \\ q \to \infty}} \delta_t^i \delta_x^\alpha v_q(x_q, t_q).$$

Now all we have to do is to show that

$$u^{i,\alpha}(x,t) = \partial_t^i \partial_x^\alpha u^{0,0}(x,t) \quad \text{on } \Omega \tag{21}$$

$\forall\, i + |\alpha| = 1$ and then, since the difference equation (16) tends to the PDE (14) as $h, k \to 0$, we will have shown that $u^{0,0}$ is a solution of the PDE. We only show (21) for i=1, but the argument is exactly the same for $|\alpha| = 1$.

Fix a value $\epsilon \geq 0$. Fix $s > 0$ such that $(x, t+s) \in \sigma$ and define $c_q \in \mathbb{N}$ the value such that $s = c_q 2^{-q}$ for large enough values of q. Since the v_q converge to $u^{0,0}$, we have for $S \ni q$ large enough,

$$\begin{aligned} &|u^{0,0}(x,t) - v_q(x,t)| < \epsilon \\ \Rightarrow &|u^{0,0}(x,t+s) - v_q(x,t+s)| < \epsilon \end{aligned}$$

Thus,

$$\Rightarrow \left| \frac{u^{0,0}(x,t+s) - u^{0,0}(x,t)}{s} - \frac{v_q(x,t+s) - v_q(x,t)}{s} \right| < \frac{2\epsilon}{s} \tag{22}$$

We also have that

$$
\begin{aligned}
& \left|\frac{v_q(x,t+s)-v_q(x,t)}{s}-\delta_0 v_q(x,t)\right| \\
= & \left|\frac{v_q(x,t+c2^{-q})-v_q(x,t)}{c2^{-q}}-\delta_0 v_q(x,t)\right| \\
= & \left|\frac{v_q(x,t+c2^{-q})-v_q(x,t+(c-1)2^{-q})}{c2^{-q}}+...+\frac{v_q(x,t+2^{-q})-v_q(x,t)}{c2^{-q}}-\delta_0 v_q(x,t)\right| \\
= & \left|\sum_{i=0}^{c-1}\frac{\delta_0 v_q(x,t+i2^{-q})}{c}-\delta_0 v_q(x,t)\right| \\
= & \left|\sum_{i=0}^{c-1}\frac{\delta_0 v_q(x,t+i2^{-q})-\delta_0 v_q(x,t)}{c}\right| \\
= & \left|\frac{2^{-q}}{c}\sum_{i=0}^{c-1}\frac{\delta_0 v_q(x,t+i2^{-q})-\delta_0 v_q(x,t+(i-1)2^{-q})}{2^{-q}}+...+\frac{\delta_0 v_q(x,t+2^{-q})-\delta_0 v_q(x,t)}{2^{-q}}\right| \\
= & \left|\frac{2^{-q}}{c}\sum_{i=0}^{c-1}\sum_{j=0}^{i-1}\delta_0^2 v_q(x,t+j2^{-q})\right| \\
\leq & \frac{2^{-q}}{c}c^2\sup_{\Omega}\left|\delta_0^2 v_q\right| \\
= & sD
\end{aligned}
\tag{23}
$$

for some constant $D<\infty$ by the estimate (19) for the second order difference quotients. Joining (22) and (23), we get

$$
\begin{aligned}
& \left|\frac{u^{0,0}(x,t+s)-u^{0,0}(x,t)}{s}-\delta_0 v_q(x,t)\right| \\
\leq & \left|\frac{u^{0,0}(x,t+s)-u^{0,0}(x,t)}{s}-\frac{v_q(x,t+s)-v_q(x,t)}{s}\right|+\left|\frac{v_q(x,t+s)-v_q(x,t)}{s}-\delta_0 v_q(x,t)\right| \\
\leq & \frac{2\epsilon}{s}+sD.
\end{aligned}
$$

Sending q to ∞ yields

$$
\left|\frac{u^{0,0}(x,t+s)-u^{0,0}(x,t)}{s}-u^{1,0}(x,t)\right|\leq\frac{2\epsilon}{s}+sD
$$

and sending ϵ to zero gives

$$
\left|\frac{u^{0,0}(x,t+s)-u^{0,0}(x,t)}{s}-u^{1,0}(x,t)\right|\leq sD
$$

and finally, sending s to zero, we have

$$\partial_t u^{0,0} = u^{1,0}$$

and similarly for all derivatives, we have proved (21), and thus the existence proof is complete.

3 Parabolic Second Order PDE

We will here use the same ideas as for the hyperbolic equation. We want to solve the general parabolic equation

$$u_t - \sum_{i=1}^{n} a_i(\vec{x},t)\partial_{x_i}^2 u - 2\sum_{i=1}^{n} b_i(\vec{x},t)\partial_{x_i} u - c(\vec{x},t)u = d(\vec{x},t) \quad \text{in } \Omega := \mathbb{R}^n \times (0,T) \tag{24}$$

$$u(x,0) = f(x) \tag{25}$$

for a function $u : \Omega = \mathbb{R}^n \times (0,T) \to \mathbb{R}$. We will assume that $a_i, b_i, c, d \in C^\infty$ uniformly bounded with uniformly bounded derivatives. We also assume that

$$\inf_{\Omega} a_j(x,t) > 0 \quad \forall j. \tag{26}$$

If we think of this last condition in the context of the heat equation, this is just assuming that the information is dissipated as we advance forward in time. The backward heat equation isn't well-posed, and we want to avoid such situations so that our scheme converges nicely. Note that by definition, any parabolic equation can be written in the form (24) with a change of variables. The general idea is once again to approximate the PDE using a discrete scheme on a grid Σ and then show that this discretized solution converges to a solution to the PDE when the grid size is sent to 0. Unless stated explicitely, we use the same notation as in section 2. We discretize the equation with central differences for the space derivatives to get the scheme

$$\begin{aligned}\frac{v(x,t+k)-v(x,t)}{k} &- \sum_{i=1}^{n} a_i(x,t)\frac{v(x+h,t)-2v(x,t)+v(x-h,t)}{h^2}\\ &- 2\sum_{i=1}^{n} b_i(x,t)\frac{v(x+h,t)-v(x-h,t)}{2h} - c(x,t)v(x,t) = d(x,t).\end{aligned} \tag{27}$$

We differ from section 2 by defining the constant

$$\lambda := \frac{k}{h^2}.$$

which will be our CFL constant for the parabolic case, as will be made clear shortly. We can write the scheme (27) explicitely in terms of $v(x,t+k)$ by using the shift operators.

We get

$$v(x,t+k) = \left(1 + \frac{k}{h^2}\sum_{j=1}^{n} a_j(E_j - 2 + E_j^{-1}) + \frac{k}{h}\sum_{i=j}^{n} b_j(E_j - E_j^{-1}) + kc\right) v(x,t) + kd(x,t)$$
$$= \sum_{j=1}^{n}\left\{(\lambda a_j + h\lambda b_j)E_j + \left(\frac{1}{n} - 2\lambda a_j + \frac{h^2\lambda}{n}c\right) + (\lambda a_j - h\lambda b_j)E_j^{-1}\right\} v(x,t) + kd(x,t). \tag{28}$$

Written in this form, we are sure that a solution to the discretized problem (27) exists $\forall (x,t) \in \Sigma$. We directly have, using the infinity norm $||\cdot|| := ||\cdot||_\infty$,

$$|v(x,t+k)| \leq \sum_{j=1}^{n}\left\{|\lambda a_j + h\lambda b_j| + \left|\frac{1}{n} - 2\lambda a_j + \frac{h^2\lambda c}{n}\right| + |\lambda a_j - h\lambda b_j|\right\} ||v|| + k||d||. \tag{29}$$

Justifying the use of this particular λ as our CFL condition, suppose we have

$$2\lambda \max_j \sup_\Omega a_j(x,t) < \frac{1}{n} \tag{30}$$

and that

$$h \max_j \sup_\Omega |b_j| < \min_j \inf_\Omega a_j. \tag{31}$$

Note that this last assumption is possible because of (26). These conditions allow us to take out some of the absolute values in the summation of (29), and we have

$$|v(x,t+k)| \leq \sum_{j=1}^{n}\left\{\lambda a_j + h\lambda |b_j| + \frac{1}{n} - 2\lambda a_j + \frac{h^2\lambda |c|}{n} + \lambda a_j - h\lambda |b_j|\right\} ||v|| + k||d||$$
$$= (1 + h^2\lambda|c|)||v|| + k||d||.$$

which gives the bound

$$||v(x,t+k)|| \leq (1 + k||c||)||v|| + k||d||. \tag{32}$$

If $||c|| = 0$, then the bound (32) is sufficient. Else, using the last result recursively for

$v(x, t = mk)$ with the initial condition (25) , we get

$$\begin{aligned}
||v(x,t+mk)|| &\le (1+k||c||)||f|| + \sum_{i=0}^{m-1}(1+k||c||)^i k||d|| \\
&= (1+k||c||)||f|| + ((1+k||c||)-1)\frac{||d||}{||c||} \\
&\le e^{mk||c||}||f|| + (e^{mk||c||}-1)\frac{||d||}{||c||} \\
&\le e^{t||c||}||f|| + te^{t||c||}||d|| \\
&\le e^{T||c||}||f|| + Te^{T||c||}||d||
\end{aligned}$$

This gives a discrete analog to the maximum principle for parabolic PDEs,

$$||v|| \le e^{T||c||}||f|| + Te^{T||c||}||d||$$

We then need similar estimates for the difference quotients of v. This task was hard for the case of hyperbolic PDEs because we only had bounds in the L_2 norm for v, and expanding these bounds to the difference quotients wasn't trivial. But here the maximum principle analog deals with the $||\cdot||_\infty$ norm,which will simplify the task. For the quotient δ_1, we can take the difference quotient of equation (28). Noting $(x+h,t) := (x_1+h, x_2, ..., x_n, t)$, we get

$$|\delta_1 v(x, t+k)|$$

$$= \left|\delta_1\left\{\sum_{j=1}^{n}\left\{(\lambda a_j + h\lambda b_j)E_j + \left(\frac{1}{n} - 2\lambda a_j + \frac{h^2\lambda}{n}c\right) + (\lambda a_j - h\lambda b_j)E_j^{-1}\right\}v(x,t) + kd(x,t)\right\}\right|$$

$$= \left|\sum_{j=1}^{n}\left\{\delta_1[(\lambda a_j + h\lambda b_j)E_j v] + \delta_1\left[\left(\frac{1}{n} - 2\lambda a_j + \frac{h^2\lambda}{n}c\right)v\right] + \delta_1[(\lambda a_j - h\lambda b_j)E_j^{-1}v]\right\} + k\delta_1 d(x,t)\right|$$

using the discrete product rule $\delta_1(pq) = (E_1 p)\delta_1 q + (\delta_1 p)q$, we get

$$\begin{aligned}
= \Bigg|\sum_{j=1}^{n} \{&E_1(\lambda a_j + h\lambda b_j)\delta_1(E_j v) + \delta_1(\lambda a_j + h\lambda b_j)E_j v \\
&+ E_1\left(\frac{1}{n} - 2\lambda a_j + \frac{h^2\lambda}{n}c\right)\delta_1 v + \delta_1\left(\frac{1}{n} - 2\lambda a_j + \frac{h^2\lambda}{n}c\right)v \\
&+ E_1(\lambda a_j - h\lambda b_j)\delta_1(E_j^{-1}v) + \delta_1(\lambda a_j - h\lambda b_j)E_j^{-1}v\Big\} + k\delta_1 d(x,t)\Bigg|
\end{aligned}$$

$$\leq \sum_{j=1}^{n} \left\{ |E_1(\lambda a_j + h\lambda b_j)| \, ||\delta_1(E_j v)|| + \left| E_1 \left(\frac{1}{n} - 2\lambda a_j + \frac{h^2 \lambda}{n} c \right) \right| \, ||\delta_1 v|| + |E_1(\lambda a_j - h\lambda b_j)| \, ||\delta_1(E_j^{-1} v)|| \right.$$

$$\left. + |\delta_1[(\lambda a_j + h\lambda b_j)| \, ||E_j v|| + \left| \delta_1 \left(\frac{1}{n} - 2\lambda a_j + \frac{h^2 \lambda}{n} c \right) \right| \, ||v|| + |\delta_1(\lambda a_j - h\lambda b_j)| \, ||E_j^{-1} v|| \right\} + k|\delta_1 d(x,t)|$$

We can ignore the space shifting operators in the norms since we are working on all of $\mathbb{R}^n$. Under the same assumptions (30) and (31), we can get rid of some absolute values to get

$$\leq \sum_{j=1}^{n} \left\{ \left| \left| \lambda a_j + h\lambda |b_j| + \frac{1}{n} - 2\lambda a_j + \frac{h^2 \lambda |c|}{n} + \lambda a_j - h\lambda |b_j| \right| \right| \, ||\delta_1 v|| \right.$$

$$\left. \left| \left| \lambda \delta_1(a_j) + h\lambda |\delta_1(b_j)| + \frac{1}{n} - 2\lambda \delta_1(a_j) + \frac{h^2 \lambda |\delta_1(c)|}{n} + \lambda \delta_1(a_j) - h\lambda |\delta_1(b_j)| \right| \right| \right\} ||v|| + k||\delta_1 d||$$

$$= (1 + k||c||) \, ||\delta_1 v|| + (1 + k||c||) \, ||v|| + k||\delta_1 d||$$

so we get the estimate

$$||\delta_1 v(t+k)|| \leq (1 + k||c||) \, ||\delta_1 v|| + (1 + k||c||) \, ||v|| + k||\delta_1 d||$$

and as we did previously, we can use this estimate recursively to get a bound on $\delta_1 v$ for any time t. The same thing can be done to bound $\delta_j^\alpha v \, \forall j = 1, ..., n$ and $|\alpha| = 1, 2, 3$. We can then take (27) and bound all the terms that relate to space shifting to get a bound on $\delta_0 v$, and then do the same thing we just did to get bounds on $\delta_0 \delta_j^\alpha \, \forall j = 1, ..., n$ and $|\alpha| = 1, 2$. Now, since everything is bounded, we can use the exact same procedure we used in section 2.2.2 to show that v converges to a solution of the PDE, thus proving existence.

It is interesting to note that this discrete construction of the solution to the parabolic PDE explicitly exhibits properties of the solution that we would expect. First, the discrete maximum principle is also valid for the limit of the v_q, and so the maximum principle applies to our constructed solution of the PDE. The maximum principle also implies uniqueness of the solution, as usual. Second, since we keep $\lambda = \dfrac{k}{h^2}$ constant as we shrink the grid size, the grid cells will get thinner and thinner with respect to time. The region of dependence of a point then tends to all of $\mathbb{R}^n$ as $h, k \to 0$, so the information for a parabolic equation travels with infinite speed.

4 Elliptic PDE

We use the same ideas for the elliptic case. We will restrict ourselves to the Laplace equation in n dimensions, but the same technique could be used for any elliptic operator satisfying the maximum principle. We want to solve the problem

$$\begin{aligned} \triangle u = f \quad \text{in } \Omega \subset \mathbb{R}^n \\ u = g \quad \text{on } \delta\Omega\,. \end{aligned} \tag{33}$$

This problem differs from the previous one because it is not a initial value problem. Such boundary value problem cannot be solved by evolving the initial condition in time, and so we need another approach. Also, since we are now working on a closed domain, we need to change our grid.

For a given grid size $h = 2^{-q}, q \in \mathbb{N}$, we define the *neighbors* $N(\vec{x})$ of a point $\vec{x} = (x_1, ..., x_n) \in \Sigma$ to be the set

$$\{(x_1 + \beta_1 h, ..., x_n + \beta_n h) \quad | \quad |\beta_i| \leq 1, \sum_{i=1}^{n} |\beta_i| = 1\}$$

We then define the grid to be

$$\Sigma_q^\Omega := \{\vec{x} \in \Sigma_q \cap \Omega \,|\, \text{all the neighbors of } \vec{x} \text{ lie inside } \Omega\}.$$

and we will define the *boundary of the grid* Γ_q to be the points in $(\Sigma_q \cap \Omega) \cap (\Sigma_q^\Omega)^c$. Such a grid structure is represented in figure 2.

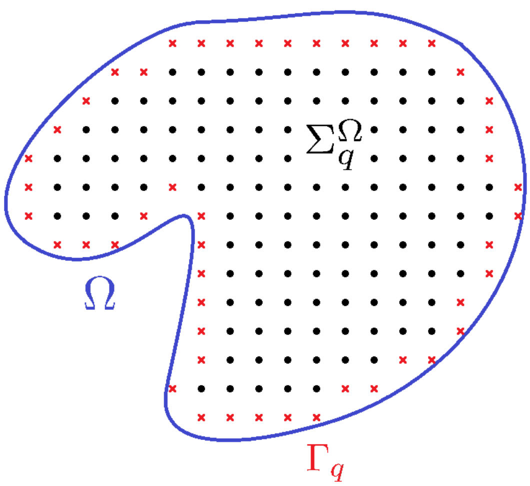

Figure 2: Grid for a certain grid size. The set Σ_q^Ω (black dots) consists of all the points of a regular grid whose neighbours all lie inside of Ω (blue curve). The boundary (red crosses) of the grid is the set of points in the grid that are inside Ω but have neighbors outside Ω.

We can discretize equation (33) using central differences. Once again, we make this choice for stability purposes. Doing so gives us the discrete scheme

$$\sum_{i=1}^{n} \frac{1}{h^2}(E_i - 2 + E_i^{-1})v = f \tag{34}$$

for points in Σ_q^Ω. For points in Γ_q , the values needed in the scheme can be defined as the value of g at this point, given that g is continuous and defined everywhere in Ω. If g in only defined on $\delta\Omega$, then take the value at the point of the boundary that is closest to the point we need to evaluate. Here, any reasonable definition of "closest" will work, as long as everything is coherent when we send the grid size to zero, and we assume that $\delta\Omega$ is smooth enough so that such a definition is possible. Doing so gives a non-homogeneous linear system with as many equations as there are unknowns, with the RHS composed of terms in g and f only, which are known. We need some properties of this scheme before proving existence of its solution v.

4.1 Maximum Principle

We show here that the discrete solution v has a property analog to the maximum principle in the continuous case, that is, for $f \geq 0$ we have

$$\max_{\Sigma_q^\Omega} v_q = \max_{\Gamma_q} v_q \,. \tag{35}$$

To show this property, we only need to look at (34) to get

$$\begin{aligned} & \sum_{i=1}^{n} \frac{1}{h^2}(E_i - 2 + E_i^{-1})v(x) \geq 0 \\ \Rightarrow \quad & v(x) \leq \frac{1}{2n} \sum_{i=1}^{n} (E_i + E_i^{-1})v(x) \\ \Rightarrow \quad & v(x) \leq \frac{1}{2n} \sum_{y \in N(x)} v(y) \\ \Rightarrow \quad & v(x) \leq \max_{y \in N(x)} v(y) \end{aligned} \tag{36}$$

And thus by using this recursively from neighbors to neighbors, we proved (35). Note also that equation (4.2) is the discrete analog of the mean value property. This maximum principle directly implies that the discretized system admits at most one equation. We thus have directly that for $f \equiv 0$ and $g \equiv 0$, the unique solution is $v \equiv 0$. And if f and g are not identically zero, the scheme (34) is only modified in its RHS, and so we know that the non-homogeneous system will also have a unique solution v since the homogeneous system has a unique solution. This proves existence and uniqueness of a solution v to the discretized system.

4.2 Bounds for the Derivatives

As we did for the other cases, we will find estimates for the difference quotients of v, but this time we use the centered differences $\bar{\delta}_i = \dfrac{E_i - E_i^{-1}}{2h}$. Once again, the maximum principle simplifies this task. More precisely, we want to show the following lemma

Lemma 4.2.1 For a point x such that the distance from this point and its neighbors to Γ is greater or equal to $R > 0$, we have

$$\left|\bar{\delta}_{x_i}\right| \leq \frac{n}{R} \max_{\Omega} |u| + \frac{R}{2} \max_{\Omega} |f| \quad \forall i \, .$$

Here f doesn't have to be zero, but we will take $g = 0$. This can be done as usual by adding the solution of the *discrete harmonic function* with the boundary data we want.

Proof. For convenience, we take $i = 1$ and $\vec{x} = 0$. Still using the infinite norm $||\cdot|| := ||\cdot||_\infty$, we consider the function

$$w(x) := \frac{||u||}{R^2}\left(\sum_{i=1}^{n} x_i^2\right) + x_1(R - x_1)\left(\frac{n||u||}{R^2} + \frac{||f||}{2}\right) .$$

We have

$$\sum_{i=1}^{n} \frac{1}{h^2}(E_i - 2 + E_i^{-1})w$$

$$= \frac{||u||}{R^2}\sum_{i=1}^{n} \frac{1}{h^2}((x_i + h)^2 - 2x_i^2 + (x_i - h)^2)$$
$$+ \left(\frac{n||u||}{R^2} + \frac{||f||}{2}\right)\frac{1}{h^2}((x_1 + h)(R - x_1 - h) - 2x_1(R - x_1) + (x_1 - h)(R - x_1 + h))$$

$$= \frac{||u||}{R^2}2n + \left(\frac{n||u||}{R^2} + \frac{||f||}{2}\right)(-2)$$
$$= -||f|| \tag{37}$$

We also have that $w(0, x_2, ..., x_n) \geq 0$, and that $w(x) \geq ||u||$ for $||x||_2 \geq 0, 0 \leq x_1 \leq R$. Define also, for points where it is possible,

$$W(x) := \frac{1}{2}(v(x_1, x_2, ..., x_n) - v(-x_1, x_2, ..., x_n))$$

then we have

$$\begin{aligned}
&\left|\left|\sum_{i=1}^{n}\frac{1}{h^2}(E_i-2+E_i^{-1})W\right|\right|\\
\leq&\frac{1}{2}\left(\left|\left|\sum_{i=1}^{n}\frac{1}{h^2}(E_i-2+E_i^{-1})(v)\right|\right|+\left|\left|\sum_{i=1}^{n}\frac{1}{h^2}(E_i-2+E_i^{-1})(v)\right|\right|\right)\\
=&||f|| \qquad (38)
\end{aligned}$$

We also have that $W(0,x_2,...,x_n)=0$ and that $|W(x)|\leq ||u||$ for $||x||_2\geq R$, $x_1\geq 0$. We can combine all these inequalities and look at the the solution on the half-ball $\{||x||_2\leq R, x_1>1\}$. Combining (37) and (38), we have that

$$\sum_{i=1}^{n}\frac{1}{h^2}(E_i-2+E_i^{-1})(w\pm W)\leq 0$$

as well as $w\pm W\geq 0$ on the discretized bountary of the half-ball. Then by using the maximum principle on this half-ball around $\vec{x}=0$, we get that $|W|\leq w$ on the half-ball, which means

$$\begin{aligned}
||\bar{\delta_1}v(0)|| &= \left|\left|\frac{1}{h}W(h,0,...,0)\right|\right|\\
&\leq \frac{1}{h}w(h,0,...,0)\\
&=\frac{1}{h}\left(\frac{||u||}{R^2}h^2+h(R-h)\left(\frac{n||u||}{R^2}+\frac{||f||}{2}\right)\right)\\
&=\frac{n||u||}{R}+\frac{R}{2}||f||+(1-n)h\frac{||u||}{R^2}\\
&\leq\frac{n||u||}{R}+\frac{R}{2}||f||
\end{aligned}$$

which proves lemma 4.2.1 for the point $\vec{x}=0$, and the complete proof follows from shifting the domain. □

Since $\bar{\delta}_i$ is also a solution of the discretized system, similar estimates can be obtained for all difference quotients of v. For instance,

$$||\bar{\delta}_i^2 v||\leq\frac{n||\bar{\delta}_i v||}{R}+\frac{R}{2}||f||\leq\frac{n}{R}\left(\frac{n||u||}{R}+\frac{R}{2}||f||\right)+\frac{R}{2}||f||\,.$$

This means that is we know *a priori* that our solution u is bounded, then all the difference quotients are bounded. All we need to do then is to send the grid size to zero and use the exact same ideas as in section 2.2.2 to show that v converges to a solution of the PDE, which concludes the proof for the elliptic case.

References

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