

The double layer potential

In this project, our goal is to explain how the Dirichlet problem for a linear elliptic partial differential equation can be converted into an integral equation by representing the answer as an integral of a fundamental solution. More precisely, by changing the double layer potential, which we will define later, to a Fredholm integral equation.

We begin with the Laplace equation in three dimensional space. The device that makes it possible for us to resolve the Dirichlet problem is the fact that we suppose that the answer can be expressed as a double layer potential of charge distributions for the surface considered. Therefore, we must make an analysis of the behavior of the potential at the boundary (because the charge is located at the boundary).

In the following, we will restrict our attention to a domain Ω which is bounded by a surface $\partial\Omega$, that is topologically equivalent to a sphere. (Opting for a sphere is simply because it is easier to treat from a mathematical point of view).

The double layer potential W is given by :

$$W(x, y, z) = \frac{1}{2\pi} \int_{\partial\Omega} \mu(\xi, \eta, \zeta) \frac{\partial}{\partial \nu} \frac{1}{r} \partial\sigma \quad (1)$$

where μ is the dipole moment density, ν is the inner normal on $\partial\Omega$ and r is given by the expression below :

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}.$$

Next, we shall compute the limits of the function W and the limits of the normal derivatives as the point (x, y, z) approaches the surface $\partial\Omega$.

Let E represent the sphere of radius R about a point (X, Y, Z) on the surface $\partial\Omega$ and let Ω^* be the union of the region Ω and the sphere E (see appendix I for the figure).

Consider the function u . For the function u in the sphere E and in a neighborhood of $\partial\Omega$, Green's identity

$$\int_{\Omega} (u\Delta v - v\Delta u) d\xi d\eta d\zeta + \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) d\sigma = 0 \quad (2)$$

means that for (x, y, z) inside the region Ω , we have :

$$\int_{\partial\Omega} \left(u \frac{\partial}{\partial \nu} \frac{1}{r} - \frac{1}{r} \frac{\partial u}{\partial \nu} \right) d\sigma = \int_{\partial\Omega^*} \left(u \frac{\partial}{\partial \nu} \frac{1}{r} - \frac{1}{r} \frac{\partial u}{\partial \nu} \right) d\sigma. \quad (3)$$

The equation (3) is going to allow us to look at the behavior of the potential W at the point (X, Y, Z) . The famous Cauchy-Kowalewski theorem makes it possible to build a local function that satisfies the following initial boundary conditions on $\partial\Omega$:

$$u = \mu, \quad \frac{\partial u}{\partial \nu} = 0 \quad (4)$$

which is a Dirichlet type boundary condition.

We have the expression below for the double layer potential W :

$$W(x, y, z) = \frac{1}{2\pi} \int_{\partial\Omega^*} \left(u \frac{\partial}{\partial \nu} \frac{1}{r} - \frac{1}{r} \frac{\partial u}{\partial \nu} \right) d\sigma \quad (5)$$

thus W must be a regular function at the point (X, Y, Z) . We can compute $W^-(X, Y, Z) = \lim_{(x,y,z) \rightarrow (X,Y,Z)} W(X, Y, Z)$, with (x, y, z) located in the interior of the region Ω (in terms of the dipole moment density μ instead of the function u by considering $x = X$, $y = Y$, $z = Z$ in formula (5) and allowing the radius R of the sphere E to approach 0).

Therefore, we have the following expression

$$W^-(X, Y, Z) = \mu(X, Y, Z) + \frac{1}{2\pi} \int_{\partial\Omega} \mu \frac{\partial}{\partial\nu} \frac{1}{r} d\sigma. \quad (6)$$

In order to evaluate the limit $W^+(X, Y, Z) = \lim_{(x,y,z) \rightarrow (X,Y,Z)} W(x, y, z)$, with (x, y, z) approaching $\partial\Omega$ from the exterior of the region Ω , we proceed in a similar manner as before and we get the following expression

$$W^+(X, Y, Z) = -\mu(X, Y, Z) + \frac{1}{2\pi} \int_{\partial\Omega} \mu \frac{\partial}{\partial\nu} \frac{1}{r} d\sigma. \quad (7)$$

In formula (7), the minus sign appears due to the orientation of the normal ν which is pointing outward of the sphere E . We can evaluate the moment density μ by subtracting W^- from W^+ . The result is

$$W^+(X, Y, Z) - W^-(X, Y, Z) = -2\mu(X, Y, Z). \quad (8)$$

Next, we will demonstrate that the normal derivative of the double layer potential W is continuous across $\partial\Omega$. Using (6) and (7), we can compute the derivatives of W^+ and W^- . We get

$$\frac{\partial W^+(X, Y, Z)}{\partial\nu} = -\frac{\partial\mu(X, Y, Z)}{\partial\nu} + \frac{1}{2\pi} \frac{\partial}{\partial\nu} \int_{\partial\Omega} \mu \frac{\partial}{\partial\nu} \frac{1}{r} d\sigma \quad (9)$$

$$\frac{\partial W^-(X, Y, Z)}{\partial \nu} = \frac{\partial \mu(X, Y, Z)}{\partial \nu} + \frac{1}{2\pi} \frac{\partial}{\partial \nu} \int_{\partial \Omega} \mu \frac{\partial}{\partial \nu} d\sigma \quad (10)$$

We subtract (10) from (9) at the point (X, Y, Z) which produces

$$\frac{\partial W^+}{\partial \nu} - \frac{\partial W^-}{\partial \nu} = -2 \frac{\partial \mu}{\partial \nu} = -2 \frac{\partial u}{\partial \nu}. \quad (11)$$

And, since the initial condition from (4) on the derivative is $\frac{\partial u}{\partial \nu} = 0$, we have

$$\frac{\partial W^+}{\partial \nu} - \frac{\partial W^-}{\partial \nu} = 0 \quad (12)$$

which implies

$$\frac{\partial W^+}{\partial \nu} = \frac{\partial W^-}{\partial \nu}.$$

The equality that comes from (12) means that the normal derivative of the double layer potential is continuous across $\partial \Omega$.

To conclude, let us solve the Dirichlet problem for the Laplace equation in $3\mathbb{D}$ by looking for a solution that has the form :

$$u(x, y, z) = W = \frac{1}{2\pi} \int_{\partial \Omega} \mu \frac{\partial}{\partial \nu} \frac{1}{r} d\sigma. \quad (13)$$

Since we wish to resolve Laplace's equation with a Dirichlet boundary condition, the initial condition must be $u = f$ on $\partial \Omega$. Using formula (6), we see that μ has to verify the equation

$$\mu(X, Y, Z) + \frac{1}{2\pi} \int_{\partial \Omega} \mu \frac{\partial}{\partial \nu} \frac{1}{r} d\sigma = f. \quad (14)$$

By the way, equation (14) has the general form

$$\phi(a) - \lambda \int k(a, b)\phi(b) db = f(a) \quad (15)$$

which is a Fredholm integral equation (or an integral equation of the second kind) where $\lambda = \pm 1$. Usually, the parameter λ and the kernel k are known. The domain Ω may or may not be bounded. We shall use equation (15) to determine the unknown function ϕ .

We will have to write formula (15) on several occasions, so let us adopt the following operator notation which is shorter and more convenient

$$K \circ \phi = \int k(a, b)\phi(b) db. \quad (16)$$

Therefore, equation (15) can now be written as

$$\phi(a) - \lambda K \circ \phi = f(a). \quad (17)$$

We can also express the iterated kernel $K^{(n)}$ as $K^{(n)} = K \circ K \circ \dots \circ K$, where K appears n times on the right-hand side.

In order to solve equation (17), we shall use the method of successive approximations. We can rewrite (17) as follows

$$\phi = f + \lambda K \circ \phi \quad (18)$$

(which is resolved for ϕ).

Then, we can substitute $\phi = f + \lambda K \circ \phi$ into (18), which produces

$$\phi = f + \lambda K \circ (f + \lambda K \circ \phi) = f + \lambda K \circ f + \lambda^2 K^{(2)} \circ \phi. \quad (19)$$

We apply the same substitution to (19) and we end up with

$$\begin{aligned} \phi &= f + \lambda K \circ f + \lambda^2 K^{(2)} \circ (f + \lambda K \circ \phi) \\ &= f + \lambda K \circ f + \lambda^2 K^{(2)} \circ f + \lambda^3 K^{(3)} \circ \phi. \end{aligned} \quad (20)$$

Repeating the same procedure leads to the Neumann series below

$$\phi = f + \lambda K \circ f + \lambda^2 K^{(2)} \circ f + \lambda^3 K^{(3)} \circ f + \dots \quad (21)$$

When the Neumann series (21) converges, it should provide a solution for equation (17). The substitution of (21) into (17) followed by a term by term integration yields the identity

$$f + \lambda K \circ f + \lambda^2 K^{(2)} \circ f + \dots - \lambda K \circ f - \lambda^2 K^{(2)} \circ f - \dots = f. \quad (22)$$

The expression given by (22) makes sense if the function f is square integrable, which means that $B^2 = \int \int (k(a, b))^2 da db < \infty$.

A simple manipulation allows us to derive a necessary condition for the function f so that a solution ϕ for equation (17) should exist. Let ψ be any non trivial solution of the homogeneous transposed equation which has the form

$$\psi - \lambda \psi \circ K = 0. \quad (23)$$

The next step consists in multiplying (17) by ψ and integrating. We get

$$\begin{aligned}\psi \circ f &= \psi \circ (\phi - \lambda K \circ \phi) \\ &= (\psi - \lambda \psi \circ K) \circ \phi\end{aligned}\tag{24}$$

From (23), we have $\psi - \lambda \psi \circ K = 0$, then $\psi \circ f = 0$ in formula (24) above, which implies that in order to solve equation (17), the necessary condition required can be expressed by saying that f has to be orthogonal to every solution ψ of the homogeneous transposed equation (23).

Finally, consider the inhomogeneous integral equation and the associated transposed equation, below :

$$\phi - \lambda K \circ \phi = f \quad (F.25) \quad \psi - \lambda \psi \circ K = g \quad (F.26)$$

and the two homogeneous equations

$$\phi - \lambda K \circ \phi = 0 \quad (F.27) \quad \psi - \lambda \psi \circ K = 0 \quad (F.28).$$

Everything that we have done thus far leads us to *the Fredholm alternative*. Either equations (F.27) and (F.28) have only the trivial solutions $\phi = \psi = 0$ or they have the same number of linearly independent eigenfunctions (non trivial solutions) $\phi_1, \phi_2, \dots, \phi_n$ and $\psi_1, \psi_2, \dots, \psi_n$. In the first case, (F.25) and (F.26) have unique solutions ϕ and ψ for every square integrable f and g . In the second one, equation (F.25) possesses a solution if and only if f is orthogonal to the n eigenfunctions ψ_i of the transposed equation $\psi_i \circ f = 0$ where $i = 1, 2, \dots, n$. While equation (F.26) has a solution if and only if g

is orthogonal to the n eigenfunctions ϕ_i of the equation $\phi_i \circ g = 0$ for $i = 1, 2, \dots, n$. More precisely, we are able to solve equation (F.25) for a given f if the homogeneous equation (F.27) has the unique solution $\phi = 0$. When we do not have uniqueness, (F.25) can be resolved if and only if the orthogonality conditions $\psi_i \circ f = 0$ for $i = 1, 2, \dots, n$ are satisfied. Let us get back to our problem. If we can find a sufficiently smooth solution μ of equation (14) using the Neumann series, the double layer potential W will represent the function needed to solve the Dirichlet problem for the Laplace equation. The double layer potential is also known as Carl Neumann's existence argument for the Dirichlet problem for the Laplace equation. It is applicable to more complicated domains (than the one featured here) and in a n dimensional space where $n \geq 3$. Finally, it is worth noting that in the literature, there exists a single layer potential, denoted V that allows one to solve the Neumann problem for the Laplace equation.

Reference

Garabedian P. R. (1964), *Partial differential equations*, John Wiley and Sons, New York, London, Sydney.

Appendix I

Appendix II

In this last section, we wish to extend the double layer potential to a n dimensional space where $n > 3$. For the most part, we will keep the notation used in the first part of the project to avoid any possible confusion.

We shall consider a domain Ω in a n dimensional space, bounded by a figure denoted $\partial\Omega$. Trying to sketch the situation in a n dimensional space would be a difficult proposition, thus we shall not do so. The double layer potential is given by the formula

$$W(x_1, \dots, x_n) = \frac{1}{2\pi} \int_{\partial\Omega} \mu(\xi_1, \dots, \xi_n) \frac{\partial}{\partial\nu} \frac{1}{d} d\sigma \quad (25)$$

where μ is the dipole moment density, ν is the inner normal and d is computed using the formula

$$d = \sqrt{(x_1 - \xi_1)^2 + \dots + (x_n - \xi_n)^2}. \quad (26)$$

At this point, the next step consists in evaluating the limits of W as well as the limits of the normal derivatives as the point (x_1, \dots, x_n) approaches the region defined by $\partial\Omega$. Consider the function u and let u be in the region E that bounds Ω and in a neighborhood of $\partial\Omega$, (here, we cannot be more specific concerning E , Ω and $\partial\Omega$ since we are in a n dimensional space). However, we can say that Ω^* denotes the union of the domain Ω and the region E . Using Green's identity

$$\int_{\Omega} (u\Delta v - v\Delta u) d\xi_1 \dots d\xi_n + \int_{\partial\Omega} \left(u \frac{\partial v}{\partial\nu} - v \frac{\partial u}{\partial\nu} \right) d\sigma = 0 \quad (27)$$

which implies that for the point (x_1, \dots, x_n) located in the region Ω , we end up with

$$\int_{\partial\Omega} \left(u \frac{\partial}{\partial \nu} \frac{1}{d} - \frac{1}{d} \frac{\partial u}{\partial \nu} \right) d\sigma = \int_{\partial\Omega^*} \left(u \frac{\partial}{\partial \nu} \frac{1}{d} - \frac{1}{d} \frac{\partial u}{\partial \nu} \right) d\sigma. \quad (28)$$

Equation (28) will be useful to explore the behavior of the potential at the point (X_1, \dots, X_n) . By using the Cauchy-Kowalewski theorem, we can construct a function which verifies the Dirichlet boundary condition below

$$u = \mu, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \quad (29)$$

The expression for W has the form

$$W(x_1, \dots, x_n) = \frac{1}{2\pi} \int_{\partial\Omega^*} \left(u \frac{\partial}{\partial \nu} \frac{1}{d} - \frac{1}{d} \frac{\partial u}{\partial \nu} \right) d\sigma. \quad (30)$$

We are in a position to evaluate

$W^-(X_1, \dots, X_n) = \lim_{(x_1, \dots, x_n) \rightarrow (X_1, \dots, X_n)} W(x_1, \dots, x_n)$ with the point (x_1, \dots, x_n) approaching $\partial\Omega$ from inside the domain Ω . Thus,

$$W^-(X_1, \dots, X_n) = \mu(X_1, \dots, X_n) + \frac{1}{2\pi} \int_{\partial\Omega} \mu \frac{\partial}{\partial \nu} \frac{1}{d} d\sigma. \quad (31)$$

We proceed similarly to calculate the value of W^+ and the result is

$$W^+(X_1, \dots, X_n) = -\mu(X_1, \dots, X_n) + \frac{1}{2\pi} \int_{\partial\Omega} \mu \frac{\partial}{\partial \nu} \frac{1}{d} d\sigma. \quad (32)$$

We evaluate the dipole moment density by subtracting (31) from (32) and we get the expression below

$$W^+(X_1, \dots, X_n) - W^-(X_1, \dots, X_n) = -2\mu(X_1, \dots, X_n) = -2u. \quad (33)$$

The last equality comes from (29) since $\mu = u$.

The next step will permit us to prove that W is continuous across $\partial\Omega$. To do so, we must compute the normal derivatives of W^+ and W^- respectively.

$$\frac{\partial W^+(X_1, \dots, X_n)}{\partial \nu} = -\frac{\partial \mu(X_1, \dots, X_n)}{\partial \nu} + \frac{1}{2\pi} \frac{\partial}{\partial \nu} \int_{\partial\Omega} \mu \frac{\partial}{\partial \nu} \frac{1}{d} d\sigma \quad (34)$$

$$\frac{\partial W^-(X_1, \dots, X_n)}{\partial \nu} = \frac{\partial \mu(X_1, \dots, X_n)}{\partial \nu} + \frac{1}{2\pi} \frac{\partial}{\partial \nu} \int_{\partial\Omega} \mu \frac{\partial}{\partial \nu} \frac{1}{d} d\sigma \quad (35)$$

By subtracting (35) from (34), we obtain the expression

$$\frac{\partial W^+}{\partial \nu} - \frac{\partial W^-}{\partial \nu} = -2 \frac{\partial \mu}{\partial \nu} = -2 \frac{\partial u}{\partial \nu} \quad (36)$$

And, since the condition (29) on the derivative implies that $\frac{\partial u}{\partial \nu} = 0$, then formula (36) becomes

$$\frac{\partial W^+}{\partial \nu} = \frac{\partial W^-}{\partial \nu}. \quad (37)$$

Hence, we have established that the potential W is continuous across $\partial\Omega$. Obviously, as one would expect, the formulas for the n dimensional space make it possible to recover the formulas for the case where $n = 3$.