THE DECOMPOSITION THEOREM AND ITS APPLICATIONS

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Abstract. This paper is devoted to proving the Decomposition Theorem for harmonic functions in $\mathbb{R}^n$. Following to that, a proof of two applications will be presented, namely, Bocher's theorem and generalization of the Laurent Series in terms of harmonic homogenous polynomials. The analysis of this paper was followed from Sheldon Axler's text on Theory of Harmonic Functions, and the technical details were thoroughly presented.

Introduction

Let $K \subset \Omega$ be compact subset of domain Ω, and let u be harmonic on $\Omega \backslash K$. Harmonic function u may be ill behaved on the boundaries $\partial\Omega$ and/or ∂K. The decomposition theorem insures that there exists a unique canonical decomposition

$$u = v + w,$$

where v is harmonic on Ω and w is harmonic on $\mathbb{R}^n \backslash K$. In other words, u may be decomposed into two harmonic functions, v and w, which are possess regularity on at least one of the boundaries. The theorem contributes to generalizing the classical Laurent series from complex analysis to $\mathbb{R}^n$. The first portion of this paper will concern those results discussed above, and the last will carry a computational proof to verify Corollary 1 which is crucial in establishing the Laurent series generalization.

Preliminary Theory

Lemma 1. [AXLER] *Suppose $K \subset \Omega$ compact. Then,*

$$\exists \varphi \in C_o^\infty(\mathbb{R}^n) \ s.t \ \varphi \equiv 1 \ on \ K,$$

$$supp \ \varphi \subset \Omega \ and \ 0 \leq \varphi \leq 1 \ on \ \mathbb{R}^n.$$

Proof. This proof is constructive. We begin by defining for $t \in \mathbb{R}$

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{1}$$

Notice that $f \in C^\infty(\mathbb{R})$. Now define for $y \in \mathbb{R}^n$

$$\Psi(y) = cf(1 - 2|y|^2), \tag{2}$$

so explicitly we have $\Psi(y) = c \cdot exp(1 - 2|y|^2)$, which then by the definition of f we have to satisfy

$$1 - 2|y|^2 > 0 \implies |y| \leq \frac{1}{\sqrt{2}}.$$

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Note that is also defines its support, hence $\Psi \in C_o^\infty(\mathbb{R}^n)$. Choosing c such that

$$1 = \int_{\mathbb{R}^n} \Psi(y)dy \implies c^{-1} = \int_{\mathbb{R}^n} exp(1-2|y|^2)dy.$$

Ψ is non-zero if $|y| \leq 1/\sqrt{2} \implies supp\ \Psi \subset B_1$ unit ball at the origin. Therefore should we define

$$\Psi_r(y) = r^{-n}\Psi(y/r), \tag{3}$$

we have,

$$\Psi_r(y) = cr^{-n}exp(1-2|y/r|^2) \quad for \quad 1-2|y/r|^2$$
$$\implies |y| \leq \frac{r}{\sqrt{2}} \tag{4}$$

$\implies supp\ \Psi_r \subset B_r$. Moreover

$$\int_{\mathbb{R}^n} \Psi_r(y)dy = 1,$$

which can be easily verified by considering a simple 2 dimensional case: Suppose

$$\iint_{\mathbb{R}^2} \phi(x,y) = 1,$$

then

$$\iint \frac{1}{r^2}\phi\left(\frac{x}{r}, \frac{y}{r}\right) dxdy$$

so by change of variables,

$$t = \tfrac{x}{r} \quad r\ dt = dx$$

$$s = \tfrac{y}{r} \quad r\ ds = dy$$

we have

$$\iint \frac{1}{r^2}\phi(t,s)r^2\ dtds = \iint_{\mathbb{R}^2} \phi(t,s) = 1.$$

Let

$$r = d(K, \partial\Omega)/3 \tag{5}$$

and define

$$\omega = \{x \in \Omega\ :\ d(x,K) < r\} \tag{6}$$

and by setting

$$\varphi(x) = \int_\omega \Psi_r(x-y)dy \tag{7}$$

for $x \in \mathbb{R}^n$, we have $\varphi \in C^\infty$ and

$$1 = \int_{\mathbb{R}^n} \Psi_r(y)dy = \int\limits_{B_r(0)} \Psi_r(y)dy \tag{8}$$

$$= \int\limits_{B_r(x)} \Psi_r(x-y)dy \geq 0 \tag{9}$$

we have $0 \leq \varphi \leq 1$ on $\mathbb{R}^n$. In particular

$$\varphi(x) = \int_\omega \Psi_r(x-y)dy = \begin{cases} 1 & x \in K, \\ 0 & d(x,K) > 2r \\ \in (0,1) & else \end{cases}$$

where condition $d(x,K) > 2r$ rises from conditions marked in expressions (4) and (5). We may now conclude that $\varphi \in C_o^\infty(\mathbb{R}^n)$. □

Theorem 1. [AXLER]*[The Decomposition Theorem]*
Let K be a compact subset of Ω. If u is harmonic on $\Omega \backslash K$, then u has a unique decomposition of the form

$$u = v + w$$

where v is harmonic on Ω and w is harmonic on $\mathbb{R}^n \backslash K$, satisfying

$$\lim_{x \to \infty} w(x) = 0, \quad for\ n > 2,$$

$$\lim_{x \to \infty} w(x) - b \log|x| = 0, \quad for\ n = 2.$$

Proof. For the $n > 2$ case: We begin with some notation: for any set $E \subset \mathbb{R}^n$ and $r > 0$, let

$$E_r = \{x \in \mathbb{R}^n \ : \ d(x,E) < r\} \tag{10}$$

Suppose first that Ω is bounded and open. Choose $r > 0$ small enough so that

$$K_r \cap (\partial\Omega)_r = \emptyset.$$

By the preceding Lemma, $\exists \varphi_r \in C_o^\infty(\mathbb{R}^n)$ supported in $\Omega \backslash K$ such that $\varphi_r \equiv 1$ on $\Omega \backslash (K_r \cup (\partial\Omega)_r)$. Now

$$u \in C^\infty(\Omega \backslash K) \implies u\varphi_r \in C_o^\infty(\mathbb{R}^n),$$

so by previous results on the fundamental solution of the Laplace equation for functions in C^2 of compact support, we have for $x \in \Omega \backslash ((\partial\Omega)_r \cup K_r)$

$$u(x) = \int\limits_{\mathbb{R}^n} \Delta_y(u\varphi_r)E(x-y)\ dy \tag{11}$$

$$= \int\limits_{(\partial\Omega)_r} \Delta_y(u\varphi_r)E(x-y)\ dy + \int\limits_{K_r} \Delta_y(u\varphi_r)E(x-y)\ dy \tag{12}$$

$$=: v_r(x) + w_r(x). \tag{13}$$

We need to show that $v_r(x)$ and $w_r(x)$ are harmonic in $\Omega \backslash (\partial\Omega)_r$ and $\mathbb{R}^n \backslash K_r$ respectively. Note that the poles of $E(x-y)$ do not lie in any of the domains of integration provided we consider the complements, allowing us to differentiate under the integral. In precise for $x \in \Omega \backslash (\partial\Omega)_r$

$$\partial_i^2 v_r(x) = \frac{\partial^2}{\partial x_i^2} \int\limits_{(\partial\Omega)_r} \Delta_y(u\varphi_r)E(x-y)\ dy \tag{14}$$

$$= \int\limits_{(\partial\Omega)_r} \Delta_y(u\varphi_r) \frac{|x-y|^2 - n(x_i - y_i)^2}{|S^{n-1}||x-y|^{n-2}} dy \tag{15}$$

and by a similar computation on $w_r(x)$ for $x \in \mathbb{R}^n \backslash K$ we have

$$\partial_i^2 w_r(x) = \int\limits_{K_r} \Delta_y(u\varphi_r) \frac{|x-y|^2 - n(x_i - y_i)^2}{|S^{n-1}||x-y|^{n-2}} dy$$

so taking sums to yield the Laplacian we have (note the exactness of the integrands)

$$\sum_i \int \Delta_y(u\varphi_r) \frac{|x-y|^2 - n(x_i - y_i)^2}{|S^{n-1}||x-y|^{n-2}} dy = \int \Delta_y(u\varphi_r) \frac{\sum_i |x-y|^2 - n\sum_i (x_i - y_i)^2}{|S^{n-1}||x-y|^{n-2}} dy$$

$$= \int \Delta_y(u\varphi_r) \frac{n|x-y|^2 - n|x-y|^2}{|S^{n-1}||x-y|^{n-2}} dy = 0,$$

concluding that $v_r(x)$ is harmonic on $\Omega\backslash(\partial\Omega)_r$ meanwhile $w_r(x)$ is harmonic on $\mathbb{R}^n\backslash K$. Moreover the limits

$$\lim_{x\to\infty} w_r(x) = \lim_{x\to\infty} \int\limits_{K_r} \Delta_y(u\varphi_r)E(x-y) = 0$$

since $E(x-y)$ vanishes at infinity, and $\varphi_r \equiv 0$ sufficiently far.
Suppose now that $s < r$, by the argument above we may decompose u into

$$u = v_s + w_s$$

on $\Omega\backslash(K_s \cup (\partial\Omega)_s)$. We claim that

$$v_r = v_s \quad on\ \Omega\backslash(\partial\Omega)_r$$

$$w_r = w_s \quad on\ (\mathbb{R}^n \cup \{\infty\})\backslash K_r$$

For $x \in \Omega\backslash(K_r \cup (\partial\Omega)_r)$ we have

$$v_r(x) + w_r(x) = v_s(x) + w_r(x)$$

$\implies$

$$v_r(x) - v_s(x) = w_s(x) - w_r(x)$$

Note that the differences agrees on the boundary so it follows that the difference $w_s - w_r$ is harmonic on the whole space. Therefore, since $w_r,\ w_s \to 0$ as $x \to \infty$, $w_s - w_r$ is bounded and harmonic on $\mathbb{R}^n$ hence by Liouville's Theorem

$$v_r - v_s = w_s - w_r \equiv 0 \implies v_r = v_s\ and w_r = w_s\ on\ \Omega\backslash(K_r \cup (\partial\Omega)_r).$$

Let $x \in \Omega$, set $v(x) = v_r(x)$ for all $r > 0$ sufficiently small so that $x \in \Omega\backslash(\partial\Omega)_r$, similarly for w and so we have

$$u = v + w$$

for bounded Ω. For unbounded Ω choose R s.t $K \subset B(0,R)$ and let

$$\omega = \Omega \cap B(0,R) \tag{16}$$

note that ω is open and follow same argument as above. $\square$

Applications.

Theorem 2. [AXLER]*[Bocher's Theorem]* $(n > 2)$ *Let* $a \in \Omega$. *If* u *is harmonic on* $\Omega\backslash\{a\}$ *and positive near* a, *then there exists harmonic function* v *on* Ω *and a nonnegative constant* b *such that*

$$u(x) = v(x) + b|x-a|^{2-n}, \quad \forall x \in \Omega\backslash\{a\}. \tag{17}$$

Proof. WLOG let $a = 0$. By Decomposition Theorem we have

$$u = v + w$$

such that $\Delta v = 0$ on Ω and $\Delta w = 0$ on $\mathbb{R}^n\backslash\{0\}$ and $\lim_{x\to\infty} w(x) = 0$. We need to show that $w(x) = b|x|^{2-n},\ b \geq 0$. We first need to ensure that $w \geq 0$: $u > 0$ and v bounded near 0, therefore $w = u - v$ is bounded near 0. Let $\epsilon > 0$ be given and set

$$h(x) = w(x) + \epsilon|x|^{2-n}$$

we have $\lim_{x\to 0} h(x) = \infty$ and $\lim_{x\to\infty} h(x) = 0$ hence by the maximum/min principal $h \geq 0$ on $\mathbb{R}^n\backslash\{0\}$. $w \to 0$ at infinity hence by the Kelvin transform:

$$K[w](x) = \begin{cases} 0 & if\ x = \infty \\ \infty & if\ x = 0 \\ |x|^{2-n} w(x/|x|^2) & if\ else \end{cases} \tag{18}$$

$$\implies |x|^{2-n}K[w] = w(x/|x|^2)\ so\ as\ x \to 0\, w(x/|x|^2) \to 0$$

since $x/|x|^2 \to \infty$. Therefore

$$K[w](x) = o(|x|^{n-2})$$

hence $K[w]$ has a removable singularity at $x = 0$. This implies that $K[w]$ extends to be nonnegative and harmonic o all of $\mathbb{R}^n$ and so by Liovilles theorem for positive harmonic functions

$$K[w] \equiv b \geq 0$$

$$\implies w(x) = b|x|^{2-n}$$

□

The Laurent Series

Definition 1. [AXLER] *Let* $\mathcal{H}_m(\mathbb{R}^n)$ *be the set of homogeneous harmonic polynomials on* $\mathbb{R}^n$ *and let* S *be the unit sphere. We define*

$$\mathcal{H}_m(S) = \{p|_S :\ p \in \mathcal{H}_m(\mathbb{R}^n)\} \tag{19}$$

where $p|_S$ *is the restriction of* p *to* S.

Corollary 1. [AXLER] *If* u *is harmonic function on a ball* $B_r(a)$, *then*

$$\exists p_m \in \mathcal{H}_m(\mathbb{R}^n)\ s.t\ u(x) = \sum_{m=0}^{\infty} p_m(x-a).$$

where the series converges absolutely and uniformly.

The proof of this is discussed at the end of this paper.

Theorem 3. [AXLER] *Let u be harmonic on the ball $B_r(a)$. Then there exists unique harmonic polynomials p_m, q_m on $\mathbb{R}^n$ such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x) + \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+n-2}}. \tag{20}$$

Proof. Suppose inner radius r and outer radius R of annulus of harmonicity of u. Then by decomposition theorem

$$u = v + w$$

where v harmonic on rB_1 and w harmonic on $\mathbb{R}^n \backslash R\overline{B_1}$ and by corollary above there exists homogenous polynomials p_m and q_m such that

$$v = (x) = \sum_{m=0}^{\infty} p_m(x)$$

harmonic on rB_1 and

$$K[w](x) = \sum_{m=0}^{\infty} q_m(x)$$

harmonic on $1/r \cdot B_1$

$$w(x) = \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+n-2}}$$

where the convergence follows from the corollary stated above. $\square$

Density of Homogenous Harmonic Polynomials in The Space of Harmonic Functions on The Ball.

The solution for the Laplace equation on the ball can be expressed by the Poisson integral,

$$u(y) = \frac{r^2 - |y|^2}{r|S^{n-1}|} \int_{S_r} \frac{u(x)}{|x-y|^n}\, dS_x$$

where S_r defined the sphere centred at y and. We may write the kernel as $P(x, y)$ so we have

$$u(y) = \int_S u(x) P(x, y)\, dS_x$$

where more formally $P(x, y)$ defined the Green's function for the Laplace on the ball. Carrying on with this notation we may write

$$P(x, y) = \frac{r^2 - |y|^2}{r|S^{n-1}|} |x-y|^{-n} \tag{21}$$

$$\textit{noting that } |x-y|^{-n} = [|x|^2 + |y|^2 - 2(x^T y)]^{-n/2} \textit{ and } |x| = r, \tag{22}$$

$$= \frac{r^2 - |y|^2}{r|S^{n-1}|} [r^2 + |y|^2 - 2(x^T y)]^{-n/2} \tag{23}$$

(24)

For convenience and without loss of generality we take $r = 1$,

$$P(x, y) = |S^{n-1}|^{-1} (1 - |y|^2)[1 + |y|^2 - 2(x^T y)]^{-n/2}.$$

Using the formal power series of $(1-z)^{-n/2}$ for $|z|<1$ we have

$$\frac{1}{(1-z)^{n/2}} = \sum_{k=0}^{\infty} c_k z^k$$

$$|S^{n-1}|P(x,y) = (1-|y|^2)[1+|y|^2-2(x^Ty)]^{-n/2} \tag{25}$$

(26) *by the power expansion above,*

$$= (1-|y|^2)\sum_{k=0}^{\infty} c_k[|y|^2+2(x^Ty)]^k \tag{27}$$

(28) *by virtue of the binomial theorem,*

$$= (1-|y|^2)\sum_{k=0}^{\infty} c_k \sum_{j=0}^{k}(-1)^j\binom{k}{j}2^{k-j}(x^Ty)^{k-j}|y|^{2j} \tag{29}$$

(30)

In the grand scheme of the following computation, it will be shown that the series representation of the Poisson kernel, may be expressed as a sum of homogeneous harmonic polynomials on the unit sphere S. Furthermore, as a result of the Weistrass approximation theorem of polynomials implies the density required to prove the corollary above.

$$P(x,y) = \sum_{k=0}^{\infty} c_k \sum_{j=0}^{k}(-1)^j\binom{k}{j}2^{k-j}(x^Ty)^{k-j}|y|^{2j} \tag{31}$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} O(p_{k+j}) \tag{32}$$

(33)

where p_{k+j} is homogeneous polynomial of degree $k+j$ in y. This can be easily verifies by inspecting $(x^Ty)^{k-j}|y|^{2j}$. For our purpose of establishing homogeneity, it is fruitful to rearrange the summation such that we obtain all homogenous polynomials of a given degree from the inner sum. That is, for every k we would like to vary j so that $j+k=m$ for a fixed m. We have

$$j+k=m \Leftrightarrow j=m-k \qquad \Longrightarrow \begin{cases} k=m & when,\ j=0 \\ 2k=m & when,\ j=k. \end{cases}$$

Keeping in mind the intent of keeping m fixed, the summation is required to be over k

$$\sum_{k=m/2}^{m} c_k(-1)^{m-k}\binom{k}{m-k}2^{2k-m}(x^Ty)^{2k-m}|y|^{2(m-k)} \tag{34}$$

Furthermore, a quick reindexing via $k \mapsto m-k$ shifts the summation from $k=m/2...m$ to $k=0...m/2$ and we have

$$\sum_{k=0}^{[m/2]} c_{m-k}(-1)^k\binom{m-k}{k}2^{m-2k}(x^Ty)^{m-2k}|y|^{2k} \tag{35}$$

Notice that $(x^T y)^{m-2k}|y|^{2k}$ is analogous to considering two function multiplication

$$\sum_{k=0}^{[m/2]} g(m-2k)f(2k) \equiv f * g(m) \tag{36}$$

which implies the homogeneity of the powers of components of y. Bring back the factor $(1-|y|^2)$ and summing over all degrees m we have

$$P(x,y) = (1-|y|^2)\sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]} c_{m-k}(-1)^k\binom{m-k}{k}2^{m-2k}(x^T y)^{m-2k}|y|^{2k} \tag{37}$$

$$= \sum_{m=0}^{\infty}\underbrace{\sum_{k=0}^{[m/2]} c_{m-k}(-1)^k\binom{m-k}{k}2^{m-2k}(x^T y)^{m-2k}|y|^{2k}}_{homogeneous\ part\ of\ degree\ m} \tag{38}$$

$$- \sum_{m=0}^{\infty}\underbrace{\sum_{k=0}^{[m/2]} c_{m-k}(-1)^k\binom{m-k}{k}2^{m-2k}(x^T y)^{m-2k}|y|^{2k+2}}_{homogeneous\ part\ of\ degree\ m+2} \tag{39}$$

(40)

Let us now consider the homogenous parts of degree m. In order to obtain a single expansion which holds all homogenous polynomials of a certain degree m, one must consider the following sum reindexing. Noticing that the second sum is of degree $m+2$ hence carrying on the following replacement, $m \mapsto m-2$ shifts the order to m, thus matching the polynomial order to m:

(41)

$$\sum_{k=0}^{[m/2]} c_{m-k}(-1)^k\binom{m-k}{k}2^{m-2k}(x^T y)^{m-2k}|y|^{2k}$$

$$-|y|^2\sum_{k=0}^{[m/2]} c_{m-2-k}(-1)^k\binom{m-2-k}{k}2^{m-2-2k}(x^T y)^{m-2-2k}|y|^{2k} \tag{42}$$

(43)

$$= \sum_{k=0}^{[m/2]} c_{m-k}(-1)^k\binom{m-k}{k}2^{m-2k}(x^T y)^{m-2k}|y|^{2k}$$

(44)

$$- |y|^2\sum_{k=0}^{[m/2]} c_{m-k-1)}(-1)^k\binom{m-2-k}{k}2^{m-2k}(x^T y)^{m-2-2k}|y|^{2k} \tag{45}$$

Explicitly we have,

$$c_k = \frac{\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+k-1)}{k!} \tag{46}$$

$$= \frac{n(n+2)\cdots(n-2k-2)}{2^k k!} \tag{47}$$

(48)

$$c_{m-k} = \frac{n(n+2)\cdots(n+2m-2k-2)}{2^{m-k}(m-k)!} \tag{49}$$

$$(-1)^k \binom{m-k}{k} c_{m-k} 2^{m-2k} = (-1)^k \frac{(m-k)!n(n+2)\cdots(n+2m-2k-2)}{k!(m-2k)!2^k(m-k)!} \tag{50}$$

$$= (-1)^k \frac{n(n+2)\cdots(n+2m-2k-2)}{2^k k!(m-2k)!}. \tag{51}$$

(52)

Meanwhile for the second part we have under the index transformation,

$$(-1)^k \frac{n(n+2)\cdots(n+2m-4-2k-2)}{2^k k!(m-2-2k)!} \tag{53}$$

Carrying on the computation one obtains

(54)

$$\sum_{k=0}^{[m/2]} (-1)^k \frac{n(n+2)\cdots(n+2m-2k-2)}{2^k k!(m-2k)!} (x^T y)^{m-2k} |y|^{2k}$$

$$- |y|^2 \sum_{k=0}^{[m/2]-1} (-1)^k \frac{n(n+2)\cdots(n+2m-4-2k-2)}{2^k k!(m-2-2k)!} (x^T y)^{m-2-2k} |y|^{2k}. \tag{55}$$

(56)

In order to combine both sums, the powers of x and y must agree. This is achieved by simple change in indices $k \mapsto k-1$ in the second sum:

(57)

$$\sum_{k=0}^{[m/2]} (-1)^k \frac{n(n+2)\cdots(n+2m-2k-2)}{2^k k!(m-2k)!} (x^T y)^{m-2k} |y|^{2k}$$

(58)

$$- |y|^2 \sum_{k=1}^{[m/2]-1+1} (-1)^{k-1} \frac{n(n+2)\cdots(n+2m-4-2(k-1)-2)}{2^{k-1}(k-1)!(m-2-2(k-1))!} (x^T y)^{m-2-2(k-1)} |y|^{2(k-1)}$$

(59)

$$= \sum_{k=0}^{[m/2]} (-1)^k \frac{n(n+2)\cdots(n+2m-2k-2)}{2^k k!(m-2k)!} (x^T y)^{m-2k} |y|^{2k}$$

(60)

$$+ \sum_{k=1}^{[m/2]} (-1)^k \frac{n(n+2)\cdots(n+2m-2k-4)}{2^{k-1}(k-1)!(m-2k)!} (x^T y)^{m-2k} |y|^{2k}$$

(61)

Note that the second sum may not be extended to $k = 0$ unless all negative factorials are adjusted. This is done by multiplying and dividing the quotient by k/k, which also in turn matches the first sums factorials:

(62)
$$\sum_{k=0}^{[m/2]}(-1)^k\frac{n(n+2)\cdots(n+2m-2k-2)}{2^k k!(m-2k)!}(x^Ty)^{m-2k}|y|^{2k}$$
(63)
$$+\sum_{k=0}^{[m/2]}(-1)^k\frac{2k\cdot n(n+2)\cdots(n+2m-2k-4)}{2^k k!(m-2k)!}(x^Ty)^{m-2k}|y|^{2k}$$
(64)
$$=\sum_{k=0}^{[m/2]}(-1)^k\frac{n(n+2)\cdots(n+2m-2k-4)}{2^k k!(m-2k)!}(x^Ty)^{m-2k}|y|^{2k}(2k+n+2m-2k-2)$$
(65)
$$=(n+2m-2)\sum_{k=0}^{[m/2]}(-1)^k\frac{n(n+2)\cdots(n+2m-2k-4)}{2^k k!(m-2k)!}(x^Ty)^{m-2k}|y|^{2k},$$

where $\sum_{k=0}^{[m/2]} O((x^Ty)^{m-2k}|y|^{2k})$ is a homogenous polynomial. Define this polynomial by Z_m. It is left to assert harmonicity of Z_m. Taking the Laplacian of Z_m, it must be shown that

$$\Delta\left[(n+2m-2)\sum_{k=0}^{[m/2]}(-1)^k\frac{n(n+2)\cdots(n+2m-2k-4)}{2^k k!(m-2k)!}(x^Ty)^{m-2k}|y|^{2k}\right]=0.$$

It would be wise to decompose this problem into subproblems using the relation

$$\Delta fg = f\Delta g + g\Delta f + 2\sum_i \partial_i f\partial_i g.$$

Calculating derivatives we have

(66) $$\frac{\partial}{\partial y_i}(x^Ty)^{m-2k}=(m-2k)(x^Ty)^{m-2k-1}x_i$$

(67) $$\frac{\partial^2}{\partial y_i^2}(x^Ty)^{m-2k}=(m-2k)(m-2k-1)(x^Ty)^{m-2k-2}x_i^2$$

(68) $$\Delta(x^Ty)^{m-2k}=(m-2k)(m-2k-1)(x^Ty)^{m-2k-2}|x|^2\ :\ |x|^2=1.$$

(69)

(70)
$$\frac{\partial}{\partial y_i}|y|^{2k}=2k|y|^{2k-1}\frac{y_i}{|y|}=2k|y|^{2k-2}y_i$$
(71)
$$\frac{\partial^2}{\partial u_i^2}|y|^{2k}=2k(2k-2)|y|^{2k-3}\frac{y_i^2}{|y|}+2k|y|^{2k-2}=2k[(2k-2)|y|^{2k-4}y_i^2+|y|^{2k-2}]$$
(72)
$$\Delta|y|^{2k}=2k[(2k-2)|y|^{2k-2}+n|y|^{2k-2}]=2k|y|^{2k-2}[2k-2+n]$$

Using the expression derived above, we return our attention to

(73) $$\Delta Z_m = |y|^{2k}(m-2k)(m-2k-1)(x^T y)^{m-2k-2}$$

(74) $$+ (x^T y)^{m-2k} 2k|y|^{2k-2}[2k-2+n]$$

(75) $$+ 2\sum_{i=1}^{n}(m-2k)(x^T y)^{m-2k-1} 2k|y|^{2k-2} x_i y_i$$

(76)

$$= (m^2-4km+4k^2+2k-m)(x^T y)^{m-2k-2}|y|^{2k} + (4k^2-4k+2kn)(x^T y)^{m-2k}|y|^{2k-2} + 4k(m-2k)(x^T y)^{m-2k}|y|^{2k-2}$$

$$= (x^T y)^{m-2k}|y|^{2k}[(m^2-4km+4k^2+2k-m)(x^T y)^{-2} + (4k^2-4k+2kn+4km-8k^2)|y|^{-2}]$$

The expression becomes

$$(n+2m-2)^{-1}\Delta Z_m$$

(77)

$$= \sum_{k=0}^{[m/2]}(-1)^k \frac{n(n+2)\cdots(n+2m-2k-4)}{2^k k!(m-2k)!}(x^T y)^{m-2k}|y|^{2k} \times$$

(78)

$$\times [(m^2-4km+4k^2+2k-m)(x^T y)^{-2} + (4k^2-4k+2kn+4km-8k^2)|y|^{-2}]$$

(79)

Define

(80)

$$Q_k = \frac{n(n+2)\cdots(n+2m-2k-4)}{2^k k!(m-2k)!}[(m^2-4km+4k^2+2k-m)(x^T y)^{-2} + (4k^2-4k+2kn+4km-8k^2)|y|^{-2}]$$

The abbreviated expression is

(81)

$$(n+2m-2)^{-1}\Delta Z_m = \sum_{k=0}^{[m/2]}(-1)^k Q_k (x^T y)^{m-2k}|y|^{2k}$$

(82) *splitting into sums over evens and odds,*

(83) $$= \sum_{k=0}^{[m/4]} Q_{2k}(x^T y)^{m-4k}|y|^{4k} - \sum_{k=0}^{[m/4]} Q_{2k+1}(x^T y)^{m-4k-2}|y|^{2k+2}$$

(84) $$= \sum_{k=0}^{[m/4]} (x^T y)^{m-4k}|y|^{4k}[Q_{2k} - Q_{2k+1}(x^T y)^{-2}|y|^2]$$

where Q_{2k+1} is adjusted so that the factorial $(m-4k-2)!$ in the denominator, is re-expressed as $(m-4k)!$. This is to ensure that the index of summation does not yield negative factorial arguments.

This concludes that the power series of the Poisson kernel $P(x, y)$ can be represented as a series in terms of homogeneous harmonic polynomials $\{Z_m\}$. Therefore, the series converges uniformly and absolutely in the ∞-norm, and $\{Z_m\}$ is dense in the set of harmonic functions on the ball.

For any harmonic functionu on the ball,

$$u(y) = \int_S u(x)P(x,y)\, dS_x \tag{85}$$

$$= \int_S u(x) \sum_{m=0}^{\infty} Z_m(x,y)\, dS_x \tag{86}$$

$$= \sum_{m=0}^{\infty} \int_S u(x) Z_m(x,y)\, dS_x. \tag{87}$$

The set such a polynomials $\{Z_m\}$, are often referred as the Zonal spherical harmonic polynomials.

Final Comment: The Orthogonality of $\{Z_m\}$ in $L^2(S)$

Definition 2 ([AXLER], p.79)**.** *For any $f, g \in L^2(S)$ square integrable on the sphere, the inner-product for Hilbert space $L^2(S)$ is defined by*

$$\langle p, q \rangle = \int_S p \cdot \overline{q}. \tag{88}$$

The following establishes the orthogonality of the Zonal harmonic polynomials on S.

Theorem 4 ([AXLER], p.79)**.** *If $p_m \in \mathcal{H}_m(\mathbb{R}^n)$ and $q_k \in \mathcal{H}_k(\mathbb{R}^n)$ with $k < m$, then*

$$\int_S p_m \cdot q_k = 0$$

Proof. By virtue of Green's Identity

$$\int_S (p_m \partial_\nu q_k - q_k \partial_\nu p_m) = \int_B (p_m \Delta q_k - q_k \Delta p_m) = 0$$

Where ν normal to the sphere. Taking ν to be radial, for any $y \in S$ we have

$$\partial_r p_m = \frac{d}{dr} p_m(ry)|_{r=1} = \frac{d}{dr} r^m p(y)|_{r=1} = m p_m(y)$$

and so similarly for q_k, we have

$$\int_S (p_m(kq_k) - q_k(mp_m)) = 0$$

$$\Leftrightarrow \ (m-k) \int_S p_m \cdot q_k = 0$$

but $m > k$ hence $\int_S p_m \cdot q_k = 0$ as required.

$\square$

References

[AXLER] Sheldon Axler, Paul Bourdon and Wade Ramey *Theory of Harmonic Functions*, Springer-Verlag NY, Inc. Second Edition (2000).

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