Asymptotics

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1 Introduction

The study of asymptotics can provide insight into certain differential equations. More precisely, say, in a D.E., there's a term involving a coefficient which is expected to be either very small or very big, we can study solutions as we let such coefficients vanish, or blow up.In turn, once we have such solutions in explicit form, we may be able to approximate solutions for when the coefficients aren't truly zero, or infinite.

This paper is split into two sections. The first treats the case of fluid flow with a perturbation. The P.D.E treating the non perturbed case is solved first, then the P.D.E. with a perturbative diffusion is solved with the method of asymptotics.

The second section introduces the WKB approximation for a nonlinear equation of second order, where the solutions are expected to be oscillatory in nature. This method can be used to solve physics related PDE's, some that come out in geometric optics or in quantum physics, namely the Schrodinger equation.

2 Fluid Flow

PDE's can be used to model fluid flow, among these we have

$$div(u\mathbf{b}) = \delta_0 \tag{1}$$

where u denotes density and **b** velocity

$$-\epsilon\Delta u^{\epsilon} + div(u^{\epsilon}\mathbf{b}) = \delta_0 \tag{2}$$

The first term in 2 denotes a diffusion term added to 1. Solving the general case of 1 can be done by first parametrizing the curve followed by the fluid.

$$\mathbf{X}(t) = \mathbf{b}(\mathbf{x}(t))\ ;\ \mathbf{X}(0) = 0 \tag{3}$$

$$\mathbf{x} = \mathbf{X}(t) + y\boldsymbol{\nu}(\mathbf{X}(t)) \tag{4}$$

In 4 $\mathbf{x}$ denotes the postion along the curve, $\boldsymbol{\nu}$ the unit normal to the curve and y the distance from the curve along $\boldsymbol{\nu}$. This permits us to express

$$\mathbf{x} = (x^1(y,t), x^2(y,t)) \tag{5}$$

$$\frac{\partial(x^1,x^2)}{\partial(t,y)} = \begin{vmatrix} \mathbf{b}^1 + y\dot{\nu^1} & \nu^1 \\ \mathbf{b} + y\dot{\nu^2} & \nu^2 \end{vmatrix} = \sigma(1-\kappa y)\; where\; \sigma = \|\mathbf{b}\| \tag{6}$$

with this we can say
$\boldsymbol{\nu} = \frac{(-b^2,b^1)}{\sigma}$ and $\dot{\boldsymbol{\nu}} = -\sigma\kappa\boldsymbol{\tau} = -\kappa\mathbf{b}$ where $\boldsymbol{\tau}$ is the unit tangent and $-\tau = \frac{\mathbf{b}}{\sigma}$

We can let the fuild density can be expressed as a density function parametrized along the curve multiplied by the dirac measure:

$$u(x) = \rho(t)\delta(y) \tag{7}$$

We can state that $\int\limits_R udx = t_2 - t_1$ between times (or the points) t_2 and t_1 on the given curve, since u denotes density

$$\int\limits_R udx = \int\limits_R \rho(t)\delta(y)\sigma(t)(1-\kappa y)dydt = \int\limits_{t_1}^{t_2} \rho(t)\sigma(t)dt = t_2 - t_1 \tag{8}$$

from this we can infer $\rho(t)\sigma(t) = 1 \Rightarrow u(x,t) = \frac{\delta(y)}{\sigma(t)}$

To confirm our result we can see that, for a generic function v(x,t) we can have

$\int D\, v\mathbf{b}\, u\, dx = \int\limits_{\mathbb{R}^2} Dv\mathbf{b}\frac{\delta(y)}{\sigma(t)}\sigma(t)(1-\kappa y)dydt = \int\limits_0^\infty Dv(\mathbf{x}(t))\cdot\mathbf{b}(\mathbf{x}(t))dt$
$= \int\limits_0^\infty \frac{d}{dt}v(\mathbf{x}(t))dt = -v(0)$

We can let the radius of the diffused dye be $O((\epsilon t)^{\frac{1}{2}})$ and we can define $z \doteq \epsilon^{-\frac{1}{2}}y$ and $v^\epsilon \doteq \epsilon^{\frac{1}{2}}u^\epsilon$. We can also express $\mathbf{b}$ as such

$$\mathbf{b} = \sigma(t)\boldsymbol{\tau} + \{\alpha(t)\boldsymbol{\tau} + \beta(t)\boldsymbol{\nu}\}y + O(y^2) \tag{9}$$

Parametrization will require the usage of the chain rule. For a generic function ω we have, in this case

$$\omega_t = \omega_{x_1}\frac{\partial x^1}{\partial t} + \omega_{x_2}\frac{\partial x^2}{\partial t} = \omega_{x_1}\sigma(1-\kappa(y))\tau^1 + \omega_{x_2}\sigma(1-\kappa(y))\tau^2 \tag{10}$$

$$\omega_y = \omega_{x_1}\frac{\partial x^1}{\partial y} + \omega_{x_2}\frac{\partial x^2}{\partial y} \tag{11}$$

or, equivalently

$$\omega_{x_1} = \frac{\omega_t\tau^2 - \omega_y\sigma(1-\kappa y)\tau^2}{\sigma(1-\kappa y)}\; and\; \omega_{x_2} = \frac{-\omega_t\tau^2 + \omega_y\sigma(1-\kappa y)\tau^1}{\sigma(1-\kappa y)} \tag{12}$$

This relation is true since $(\tau^1\nu^2 - \nu^1\tau^2) = 1$ and since ν is the unit normal to the curve and τ is the unit tangent to the curve. Using 9, we have

$$\mathbf{b}\cdot Du^{\epsilon} = (\sigma(t)\tau + (\alpha(t)\tau + \beta(t)\nu y + O(y^2))\cdot(u_{x_1}\hat{x_1} + u_{x_2}\hat{x_2}) \tag{13}$$

$$\boldsymbol{\tau} = \frac{\frac{\partial x^1}{\partial t}\hat{x_1} + \frac{\partial x^2}{\partial t}\hat{x_2}}{\left\|(\frac{\partial x^1}{\partial t}\hat{x_1})^2 + (\frac{\partial x^2}{\partial t}\hat{x_2})^2\right\|} = \frac{\sigma(1-\kappa y)(\tau_1\hat{x_1} + \tau_2\hat{x_2})}{\left\|(y\sigma(1-\kappa y))^2 + ({\tau_1}^2 + {\tau_2}^2)\right\|} = \tau_1\hat{x}_1 + \tau_2\hat{x}_2 \tag{14}$$

The last equality holds since ${\tau^2}_1 + {\tau^2}_2 = 1$. It was not a priori evident, but comes out naturally as a result of the parametrization, as 10 and 11 were used.

Similarly, we can obtain

$$\boldsymbol{\nu} = \nu_1\hat{x_1} + \nu_2\hat{x_2} \tag{15}$$

And so, using $12, 13, 14, 15$ we have

$$\mathbf{b}\cdot Du^{\epsilon} = \frac{{u_t}^{\epsilon}}{1-\kappa y} + \frac{\alpha y u_t^{\epsilon}}{\sigma(1-\kappa y)} + \beta y {u_y}^2 + O(y^2 \mid Du^{\epsilon} \mid) \tag{16}$$

since $v^{\epsilon} = \epsilon^{1/2}u^{\epsilon}$ and $z = \epsilon^{\frac{-1}{2}}y$ we now have, for $\epsilon \ll$

$$\frac{{u_t}^{\epsilon}}{1-\kappa y} \simeq \epsilon^{-1/2}{v_t}^{\epsilon}\,;\ \frac{\alpha y u_t^{\epsilon}}{\sigma(1-\kappa y)} \simeq \frac{\alpha z \epsilon^{\frac{1}{2}} u_t^{\epsilon}}{\sigma(1-\kappa z\epsilon^{\frac{1}{2}})} \to o(\epsilon^{1/2}) \tag{17}$$

$$u_y = \frac{1}{\epsilon}{v^{\epsilon}}_z \Rightarrow \beta y {u_y}^{\epsilon} = \beta\epsilon^{-\frac{1}{2}} z {v^{\epsilon}}_z\,;\ O(y^2 \mid Du^{\epsilon} \mid) \to O(\epsilon^{\frac{1}{2}}) \tag{18}$$

and so, substituting these previous equations into 16 we end up with

$$\mathbf{b}\cdot Dv^{\epsilon} = {v_t}^{\epsilon} + \beta z {v_z}^{\epsilon} + O(\epsilon^{\frac{1}{2}}) \tag{19}$$

Similarly, using 9, we have
$div\mathbf{b} = \frac{1}{\sigma(1-\kappa y)}(b_t^1\nu^2 - b_t^2\nu^1) + (\tau^1 b_y^2 - \tau^2 b_y^1)$
or

$$div\mathbf{b} = \frac{1}{\sigma(1-\kappa y)}((\sigma\tau^1)_t\nu^2 - (\sigma\tau^2)_t\nu^1) + (\tau^1(\alpha\tau^2 + \beta\nu^2) - \tau^2(\alpha\tau^1 + \beta\nu^1)) + O(y) \tag{20}$$

The second term in 20 can be identified as β. From the first term we can see that:

$$((\sigma\tau^1)_t\nu^2 - (\sigma\tau^2)_t\nu^1) = \sigma\tau_t^1\nu^2 - \sigma\tau_t^2\nu^1 + \dot{\sigma}(\tau^1\nu^2 - \tau^2\nu^1) = \sigma^2\kappa(\nu^1\nu^2 - \nu^2\nu^1) + \dot{\sigma}(1) = \dot{\sigma} \tag{21}$$

from this we have, recalling that we can express y as $z\epsilon^{\frac{1}{2}}$

$$div\mathbf{b} = (\frac{\dot{\sigma}}{\sigma} + \beta) + O(y)\ as\ (1-\kappa y) \simeq 1\ for\ \epsilon \ll \tag{22}$$

$$\epsilon\Delta u^{\epsilon} = \epsilon(\partial_{yy}v^{\epsilon} + \partial_{tt}v^{\epsilon}) = \epsilon(\epsilon^{-1}\partial_{zz}v^{\epsilon} + \partial_{tt}v^{\epsilon}) = \partial_{zz}v^{\epsilon} + o(\epsilon^{\frac{1}{2}}) \tag{23}$$

Going back to the original differential equation, we have

$$-\epsilon\Delta u^\epsilon + div(u^\epsilon \mathbf{b}) = \delta_0 = -\partial_{zz}v^\epsilon + o(\epsilon^{\frac{1}{2}}) + v^\epsilon\nabla\cdot\mathbf{b} + \mathbf{b}\nabla v^\epsilon \tag{24}$$

Equation 24 can be rewritten as

$$-v^\epsilon_{zz} + o(\epsilon^{\frac{1}{2}}) + (\frac{\dot{\sigma}}{\sigma} + \beta)v^\epsilon + v^\epsilon_t + \beta z v^\epsilon_z + O(\epsilon^{\frac{1}{2}}) \tag{25}$$

as $\epsilon \to 0$, we have $v^\epsilon \to v$ in $\mathbb{R}^2$ which lets us cleverly write, in order to solve v:

$$v_t - v_{zz} + (\beta z v)_z + \frac{\dot{\sigma}}{\sigma}v = O(\epsilon^{1/2})\; in\, \mathbb{R}\times(0,\infty) \tag{26}$$

This equation can be solved analytically. Once we have solved for v with a given initial condition, of the form $v(t=0) = (\frac{\delta z}{\sigma(0)})$, we can find u:

$$u^\epsilon = \epsilon^{-\frac{1}{2}}v^\epsilon = \epsilon^{-\frac{1}{2}}(v + o(1)) \tag{27}$$

3 WKB Method for Non-Linear Equations

3.1 General treatment

Consider

$$\epsilon^2\ddot{x} + f(t,x) = 0 \tag{28}$$

with $\epsilon > 0, f \in C^\infty(\mathbb{R})$ with an ϵ dimensionfull. Here let's assume the solution for x to be oscillatory in nature as $\epsilon \to 0$.

It is not uncommon for such equations to arise in physical systems. For example, say for a potential $U(x) = \int_0^x f(y)dy$ we have that the energy $E = \frac{1}{2}\epsilon^2\dot{x}^2 + U(x)$ is conserved in time, taking a derivative with respect to time leads back to a certain form of 28. Furthermore, the WKB method is widely used to solve certain forms of the Schrodinger equation. More precisely, regions where the total energy is larger the the potential energy are called classically allowed regions since the kinetic energy is positive. Solutions to the Schrodinger equation in a classically allowed region are expected to be oscillatory in nature, and so the WKB method is a great tool, especially when potentials take a really strange shape which can make the Schrodinger equation difficult, if not impossible to solve analytically.

Now to begin, we can treat the case where $f(x,t) = f(x)$

$$\epsilon^2\ddot{x} + f(x) = 0 \tag{29}$$

Say we let $y(t_1,\epsilon) = x(t,\epsilon)$, let $t_1 = \frac{t}{\epsilon}$ we then have

$$\frac{d^2y}{dt_1^2} + f(y) = 0 \tag{30}$$

This gives us motivation, while considering the more general equation 28, to let

$$y(t_1,t,\epsilon) = x(t,\epsilon) = \; where\, t_1 = \frac{S(t)}{\epsilon} \tag{31}$$

Using chain rule we can compute the following:

$$\dot{x} = \frac{\partial y}{\partial t_1}\frac{\partial t_1}{\partial t} + \frac{\partial y}{\partial t} \Rightarrow \frac{\partial}{\partial t}(\frac{\partial y}{\partial t_1}(t, t_1\epsilon))\frac{\partial t_1}{\partial t}ect = \frac{\partial^2 y}{\partial t_1^2}(\frac{\partial t_1}{\partial t})^2 + \frac{\partial^2 y}{\partial t_1 \partial t}\frac{\partial t_1}{\partial t} + \frac{\partial y}{\partial t_1}\frac{\partial^2 t_1}{\partial t^2} \tag{32}$$

$$\ddot{x} = \frac{\partial^2 y}{\partial t_1^2}\frac{\partial^2 t_1}{\partial t^2} + 2\frac{\partial^2 y}{\partial t_1 t}\frac{\partial t_1}{\partial t} + \frac{\partial y}{\partial t_1}(\frac{\partial t_1}{\partial t})^2 + \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial t_1^2}\frac{\ddot{S}}{\epsilon} + 2\frac{\partial^2 y}{\partial t_1 \partial t}\frac{\dot{S}}{\epsilon} + \frac{\partial y}{\partial t_1}\frac{\partial \dot{S}^2}{\epsilon^2} + \frac{\partial^2 y}{\partial t^2} \tag{33}$$

or

$$\epsilon^2 \ddot{x} = \dot{S}^2 \frac{\partial^2 y}{\partial t_1^2} + \epsilon L_1 y + \epsilon^2 \frac{\partial^2 y}{\partial t^2} where L_1 = 2\dot{S}\frac{\partial^2}{\partial t_1 \partial t} + \ddot{S}\frac{\partial}{\partial t_1} \tag{34}$$

Now we can let $y = \sum_{n=0}^{\infty} y_n(t_1, t)\epsilon^n$ where ϵ is dimensionfull.

$$\dot{S}^2 \frac{\partial^2 y}{\partial t_1^2} + \epsilon L_1 y + \epsilon^2 \frac{\partial^2 y}{\partial t^2} + f(t, y_0 + \epsilon y_1 + \epsilon^2 y_2 + ...) = 0 \tag{35}$$

$$\dot{S}^2 \frac{\partial^2}{\partial t_1^2}(y_0 + y_1\epsilon + O) + \epsilon L_1(y + y_1\epsilon + O) + \epsilon^2 \frac{\partial^2}{\partial t^2}(y + y_1\epsilon + O) + f(t, y_0) + \epsilon y_1 f_x(t, y_0) + O \tag{36}$$

The beauty of expanding around ϵ is that this is a term that is dimensionfull, and so by dimensional analysis we can gather terms with the same order of ϵ:

$$\epsilon^0 terms : \dot{S}^2 \frac{\partial^2}{\partial t_1^2} y_0 + f(t, y_0) = 0 \tag{37}$$

$$\epsilon^1 terms : \epsilon \dot{S}^2 \frac{\partial^2 y_1}{\partial t_1^2} + \epsilon L_1 y_0 + \epsilon y_1 f_x(t, y_0) = 0 \tag{38}$$

Evidently, we can do the same for terms of higher order ϵ^n. Now let's assume $y_0(t_1, t)$ is periodic in t_1 by $mT; m \in \mathbb{N}$, with period $T = T(t_1)$ and not $T(t)$.

Applying $\frac{\partial}{\partial t_1}$ to 37 we obtain

$$\dot{S}^2 \frac{\partial^3}{\partial t_1^3} y_0 + f_x(t, y_0)\frac{\partial y_0}{\partial t_1} = 0 \tag{39}$$

It is useful to define

$$L_0 = \dot{S}^2 \frac{\partial^2}{\partial t_1^2} y_0 + f_x(t, y_0) \tag{40}$$

this allows us to state, using 38

$$-L_0 y_1 = L_1 y_0 \tag{41}$$

It is useful to first find the solution to the homogenous D.E. $L_0\omega = 0$ by equating it to 39

$$L_0\omega = 0 = \dot{S}^2\frac{\partial^3}{\partial t_1^3}y_0 + f_x(t, y_0)\frac{\partial y_0}{\partial t_1} = 0 \Rightarrow \omega_1(t_1, t) = \frac{\partial}{\partial t_1}y_0(t_1, t) \tag{42}$$

Now for the solution for 41 we know that the solution will be the sum of the homogeneous and particular solutions, that are, by construction, both orthogonal. By assumption, both L_1y_0 and ω_1 are periodic. It is thus fair to state:

$$0 = \int_0^T ((\ddot{S}\frac{\partial}{\partial t_1} + 2\dot{S}\frac{\partial^2}{\partial t_1 \partial t})y_0)\frac{\partial y_0}{\partial t_1}dt_1 = \int_0^T \ddot{S}(\frac{\partial y_0}{\partial t_1})^2 + \int_0^T \dot{S}2\frac{\partial^2 y_0}{\partial t_1 \partial t}\frac{\partial y_0}{\partial t_1}dt_1 \tag{43}$$

this in turns implies

$$0 = \frac{\partial}{\partial t}\dot{S}\int_0^T (\frac{\partial y_0}{\partial t_1})^2 dt_1 = 0 \tag{44}$$

$$\dot{S}\int_0^T (\frac{\partial y_0}{\partial t_1})^2 dt_1 = c_0 \tag{45}$$

3.2 Example of WKB treatment

Example 1 : $\epsilon^2\ddot{x} + a(t)x = 0$ where $a(t) > 0$ where we expect a T periodic solution By inspection, we see that $f(x, t) = a(t)x$ from this, we can use 37 to express

$$\dot{S}^2\frac{\partial^2 y_0}{\partial t_1^2} + a(t)y_0 = 0 \tag{46}$$

Which leads us to the solution

$$y_0(t_1, t) = A(t)cos(\frac{\sqrt{a}}{\dot{S}}t_1) \tag{47}$$

from expected periodicity of the equation we have

$$\sqrt{a}\frac{T}{\dot{S}} = 2\pi n \Rightarrow \dot{S}(t) = T\frac{\sqrt{a}}{2\pi n} \tag{48}$$

$$\Rightarrow S(t) = \frac{T}{2\pi n}\int_0^t \sqrt{a(\tau)}d\tau \tag{49}$$

We can set $S(t) = \int_0^t \sqrt{a(\tau)}d\tau; T = 2\pi;\ n = 1 \Rightarrow \frac{\sqrt{a}}{\dot{S}} = 1$

In 47 we can replace $\frac{\sqrt{a}}{\dot{S}}$ by 1, and t_1 by $S(t)\epsilon^{-1}$ where S is of the form expressed in 49

$$y_0 = A(t)cos\{\frac{1}{\epsilon}\int_0^t \sqrt{a(\tau)}d\tau\} \tag{50}$$

From 45 we have

$\dot{S}A^2(t)\int_0^{2\pi} sin^2(t_1)dt_1 = C \Rightarrow \sqrt{a}A^2(t)\pi = C \Rightarrow A(t) = Ca^{\frac{-1}{4}}(t)$

And so, constants omitted, we have

$$x_1(t,\epsilon) = a^{-1/4}(t)cos\{\frac{1}{\epsilon}\int_0^t \sqrt{a(\tau)}d\tau\} \tag{51}$$

similarly, we have

$$x_2(t,\epsilon) = a^{-1/4}(t)sin\{\frac{1}{\epsilon}\int_0^t \sqrt{a(\tau)}d\tau\} \tag{52}$$

4 Bibliography

L. Craig Evans, Partial differential equations. AMS 1998.
V Fedoryuk, Partial Differential Equations. Springer.