MATH 580 ASSIGNMENT 5

DUE TUESDAY NOVEMBER 22

1. Let Ω be a bounded domain with $C^{k+2,\alpha}$ boundary, and let L be a second order linear elliptic operator with $C^{k,\alpha}(\bar{\Omega})$ coefficients.
   (a) Prove the following *Schauder estimate*
   $$\|u\|_{C^{k+2,\alpha}(\Omega)} \lesssim \|Lu\|_{C^{k,\alpha}(\Omega)} + \|u\|_{C(\Omega)}, \qquad u \in C^{k+2,\alpha}(\bar{\Omega}).$$
   The $k = 0$ case is treated in class, which can be assumed.
   (b) Show that if $u \in C^{2,\alpha}(\bar{\Omega})$ satisfies
   $$Lu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$
   with $f \in C^{k,\alpha}(\bar{\Omega})$, then $u \in C^{k+2,\alpha}(\bar{\Omega})$. You may assume that the lowest order coefficient of L is so that the maximum principle holds for L, but also try without this assumption.
2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary, and consider the nonlinear Dirichlet problem
   $$\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 1 \quad \text{on } \partial\Omega, \tag{1}$$
   where $f : I \to \mathbb{R}$ is a sufficiently smooth function defined on some interval $I \subseteq \mathbb{R}$. Then we look for a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$, with $u(x) \in I$ for $x \in \bar{\Omega}$. The choice of I may depend on the nature of f, or on the context of the problem. For example, if f is given by $f(u) = u^{-1}$, then a natural choice would be $I = (0, \infty)$. This would also be the choice if one is interested in finding only a positive solution u.
   (a) Consider the case $f(u) = u^m$ with $m \in \mathbb{N}$ odd. Show that any solution of (1) in $C^2(\Omega) \cap C(\bar{\Omega})$ must satisfy $0 \leq u \leq 1$ in $\bar{\Omega}$, and is unique.
   (b) Show that the only solution of (1) with $f(u) = u - u^{-1}$ is $u \equiv 1$.
3. We shall establish the existence of a solution to (1) by the so-called *sub-supersolution method*. To this end, a function $u_- \in C^2(\Omega) \cap C(\bar{\Omega})$ with $u_-(\bar{\Omega}) \subset I$ is called a *subsolution* to the above problem if
   $$\Delta u_- \geq f(u_-) \quad \text{in } \Omega, \qquad u_- \leq 1 \quad \text{on } \partial\Omega.$$
   Similarly, a function $u_+ \in C^2(\Omega) \cap C(\bar{\Omega})$ with $u_+(\bar{\Omega}) \subset I$ is a *supersolution* if
   $$\Delta u_+ \leq f(u_+) \quad \text{in } \Omega, \qquad u_+ \geq 1 \quad \text{on } \partial\Omega.$$
   (a) Construct sub- and supersolutions satisfying $u_- \leq u_+$ in $\bar{\Omega}$, for the case $f(u) = \alpha u^m - \beta u^{-k}$ with $m, k \in \mathbb{N}$ and $\alpha, \beta \geq 0$. If $\beta \neq 0$ choose $I = (0, \infty)$.

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(b) Let u_- and u_+ be sub- and supersolutions satisfying $u_- \le u_+$ in $\bar\Omega$, and let $a = \min u_-$ and $b = \max u_+$. Choose $\lambda \ge 0$ so that $s \mapsto f(s) - \lambda s$ is non-increasing on the interval $[a, b]$. Show that such a choice is possible. Let the sequence $u_k \in C^2(\Omega) \cap C(\bar\Omega)$, $(k = 0, 1, \ldots)$, be defined by $u_0 = u_+$ and

$$\Delta u_k - \lambda u_k = f(u_{k-1}) - \lambda u_{k-1} \quad \text{in } \Omega, \qquad u_k = 1 \quad \text{on } \partial\Omega,$$

for $k \in \mathbb{N}$. Justify the existence of this sequence, and show that

$$u_- \le u_k \le u_{k-1} \le u_+ \quad \text{in } \bar\Omega,$$

for all $k \in \mathbb{N}$.

(c) By using, for example, the estimate

$$\|u_k\|_{C^1(\Omega)} \lesssim \|f(u_{k-1})\|_{C(\Omega)} + \|u_{k-1}\|_{C(\Omega)} + 1,$$

and a compactness argument, show that the sequence $\{u_k\}$ from (b) converges uniformly in $\bar\Omega$ to a function $u \in C(\bar\Omega)$. Note that the above estimate is easy to get from the potential (or Schauder) estimates we proved in class.

(d) Update the uniform convergence of $u_k \to u$ to a C^1 convergence, i.e., show that $\|u_k - u\|_{C^1(\Omega)} \to 0$ as $k \to \infty$. With the help of the Schauder estimates, further update it to a $C^{2,\alpha}$ convergence.

(e) Prove that u is a solution of (1).

(f) Provide a new example of f that can be treated by this method. In particular, construct sub- and supersolutions for your example. How do we modify the method if we want to handle the general Dirichlet condition $u = g$ on $\partial\Omega$?

4. Prove that if g is a bounded continuous function on $\mathbb{R}^n$, then

$$e^{t\Delta} e^{s\Delta} g = e^{(t+s)\Delta} g,$$

for $s, t > 0$. In combination with the property $e^{t\Delta} g \to g$ as $t \to 0$, this means that the heat propagators $e^{t\Delta}$, $(t > 0)$, form a *one-parameter semigroup* of operators.

5. Using the heat kernel, devise an approach analogous to Green's formula (and/or the Green function approach) for representing solutions of the heat equation on a bounded spatial domain $\Omega \subset \mathbb{R}^n$ and a bounded time interval $(0, T)$.

6. By way of examples, make a strong case against the well-posedness of the Cauchy problem for the *backward heat equation*

$$\partial_t u + \Delta u = 0 \quad \text{in } \{t > 0\}, \qquad u = g \quad \text{on } \{t = 0\},$$

or equivalently, of the *backward Cauchy problem* for the heat equation

$$\partial_t u = \Delta u \quad \text{in } \{t < 0\}, \qquad u = g \quad \text{on } \{t = 0\}.$$