MATH 580 ASSIGNMENT 3

DUE THURSDAY OCTOBER 13

1. Show that
$$\partial_i|x| = x_i/|x|, \qquad \text{where} \qquad |x| = \sqrt{x_1^2 + \ldots + x_n^2}.$$
Let $u(x) = \phi(|x|)$, with a twice differentiable function $\phi : (a, b) \to \mathbb{R}$. Then prove that
$$\Delta u(x) = \phi''(|x|) + \frac{n-1}{|x|}\phi'(|x|), \qquad \text{for} \qquad a < |x| < b.$$
Find all solutions of $\Delta u = 0$, where u is of the above form with $(a, b) = (0, \infty)$.
2. In $\mathbb{R}^n$, define the *Kelvin transform* for functions by
$$(Ku)(x) = |x|^{2-n}u(x/|x|^2).$$
Show that if u is harmonic in $\Omega \subset \mathbb{R}^n$, then Ku is harmonic in
$$\Omega' = \{x/|x|^2 : x \in \Omega \setminus \{0\}\}.$$
What is Ω' if $\Omega = \{x \in \mathbb{R}^n : x_n > 1\}$? Show that $K^{-1} = K$, i.e., that K is an involution.
3. Let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ for some $r > 0$, and let g be a continuous function on the boundary ∂B_r. Let $\sigma_{n-1} = |\partial B_1|$ be the $(n-1)$ dimensional volume of the unit sphere in $\mathbb{R}^n$. Show that the function
$$u(x) = \frac{r^2 - |x|^2}{\sigma_{n-1} r} \int_{\partial B_r} \frac{g(y)\mathrm{d}S_y}{|x-y|^n}, \qquad (x \in B_r),$$
satisfies $u \in C^2(B_r)$, and that $\Delta u = 0$ in B_r and $u(x) \to g(z)$ as $B_r \ni x \to z \in \partial B_r$. You can restrict to the radial limit, i.e., the case $x = zt$ with $(0, 1) \ni t \to 1$. What would be the analogous formula for $n = 1$?
4. Let u be a harmonic function in $\mathbb{R}^n$ satisfying $u(x) \geq -C(1 + |x|)^m$ for some constants C and $m \in \mathbb{N}$. Show that u is a polynomial of degree less or equal to m.
5. Consider the equation
$$(\Delta + \lambda)u = 0,$$
in a bounded domain $\Omega \subset \mathbb{R}^n$, where λ is a real parameter. We say that the maximum principle holds for the particular value λ if $(\Delta + \lambda)u = 0$ in Ω implies
$$u(x) \leq \sup_{\partial\Omega} u, \qquad \text{for all} \quad x \in \Omega.$$
Try to identify the set of $\lambda \in \mathbb{R}$ for which the maximum principle holds by proving the maximum principle for some values of λ and exhibiting counterexamples for other values.

Date: Fall 2011.

6. Let Ω be an open subset of $\mathbb{R}^n$. Show that if $u \in C^2(\Omega)$ is harmonic in Ω then
$$\int_{\partial B} \partial_\nu u = 0,$$
for any ball B whose closure is contained in Ω. Here $\partial_\nu u$ is the normal derivative of u. Conversely, prove that if $u \in C^1(\Omega)$ satisfies the above property for any ball B whose closure is contained in Ω, then u is harmonic in Ω.