MATH 580 ASSIGNMENT 2

DUE THURSDAY SEPTEMBER 29

1. For each of the following cases, determine the characteristic cones and characteristic surfaces.
   a) Wave equation with wave speed $c > 0$: $u_{xx} + u_{yy} = c^{-2} u_{tt}$.
   b) Tricomi-type equation: $u_{xx} + y u_{yy} = 0$.
   c) Ultrahyperbolic "wave" equation: $u_{xx} + u_{yy} = u_{zz} + u_{tt}$.
2. Consider the Cauchy problem for the Laplace equation:
$$u_{xx} + u_{tt} = 0, \qquad u(x,0) = 0, \qquad u_t(x,0) = \phi(x).$$
   For given $\varepsilon > 0$ and an integer $k > 0$, construct an initial datum ϕ such that
$$\|\phi\|_\infty + \ldots + \|\phi^{(k)}\|_\infty < \varepsilon,$$
   and
$$\|u(\cdot, \varepsilon)\|_\infty > 1/\varepsilon.$$
3. Solve
$$x u_x + 2y u_y + u_z = 3u, \qquad u(x,y,0) = g(x,y).$$
4. Prove that if $\beta \in \mathbb{R}$ and $u \in C^1(\mathbb{R}^2)$ is a solution of $u_t + \beta u_x = 0$, then
$$\{(x,t) : u \in C^k \text{ on a neighbourhood of } (x,t)\},$$
   is a union of rays.
5. Consider the equation
$$xy u_x + (2y^2 - x^6) u_y = 0, \qquad x > 0,\ y > 0.$$
   Determine and sketch the characteristics. For $n \in \mathbb{N}$ and $\alpha > 0$, consider the initial condition
$$u(x, \alpha x^n) = x^2.$$
   For which $\alpha > 0$ does the problem have a solution? Give an explicit expression for the solution. For which $\alpha > 0$ is the solution uniquely determined? Answers may depend on n (Try $n = 1$ and $n = 2$ etc first).
6. Consider the initial value problem
$$u_t + u u_x + \gamma u = 0, \qquad u(x,0) = f(x),$$
   for $(x,t) \in \mathbb{R} \times [0,\infty)$, where $\gamma \geq 0$ is a constant. Make a rough sketch of the characteristics on an xt diagram and show that wave breaking occurs only if $f'(x) < -\gamma$ for some x.

Date: Fall 2011.

7. Consider the Burgers equation $u_t + uu_x = 0$ with initial data

$$u(x,0) = 0 \text{ if } |x| \geq 1 \qquad \textit{and} \qquad u(x,0) = 1 - |x| \text{ if } |x| \leq 1.$$

By sketching the characteristics, describe the entropy solution. Clearly indicate on your sketch of the characteristics where the shock is. Is the shock a line? What is the equation of the shock? What happens to $u(\cdot, t)$ as $t \to \infty$?