Sobolev Space Through the Bessel Potential

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December 14, 2011

1 Introduction

This paper will go about the proof of the *Sobolev embedding theorem.* We will make use the following definitions.

Definition $\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$

Definition $\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C_c^\infty(\mathbb{R}^n) \colon \sup |x^\beta \partial^\alpha f| < \infty, \forall \text{ multi-indices } \alpha, \beta \right\}$

Definition $\phi_j \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ if for all multi-indices α and β we have $x^\beta \partial^\alpha \phi_j \to 0$ uniformly on $\mathbb{R}^n$.

Definition $\mathcal{S}^*(\mathbb{R}^n)$ is the set of sequentially continuous linear functionals on the space $\mathcal{S}(\mathbb{R}^n)$.

2 The Bessel Potential

Definition For $s \in \mathbb{R}$, we define the *Bessel potential of order s* to be the (sequentially) continuous bijective linear operator $\mathcal{J}^s \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{J}^s u = \mathcal{F}^{-1} \left(1 + |\cdot|^2\right)^{s/2} \mathcal{F} u$$

Notice for $s, t \in \mathbb{R}$,

$$\mathcal{J}^{s+t} = \mathcal{J}^s \mathcal{J}^t, \quad (\mathcal{J}^s)^{-1} = \mathcal{J}^{-s}, \quad \mathcal{J}^0 = I$$

In addition, from Plancherel's theorem we have

$$(\mathcal{J}^s u, v)_{L^2(\mathbb{R}^n)} = (u, \mathcal{J}^s v)_{L^2(\mathbb{R}^n)}$$

for all $u, v \in \mathcal{S}(\mathbb{R}^n)$, which motivates a natural extension of $\mathcal{J}^s \colon \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$ defined by

$$\langle \mathcal{J}^s u, \phi \rangle = \langle u, \mathcal{J}^s \phi \rangle, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

Definition For any $s \in \mathbb{R}$, we define the *Sobolev space of order* s on $\mathbb{R}^n$, denoted $H^s(\mathbb{R}^n)$, by

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}^*(\mathbb{R}^n) : \mathcal{J}^s u \in L^2(\mathbb{R}^n) \right\}$$

Remark For $w \in \mathcal{S}^*(\mathbb{R}^n)$, we write $w \in A$ when $\exists\, g \in A$ such that

$$\langle w, \phi \rangle = (g, \phi)_{L^2(\mathbb{R}^n)}, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$$

$H^s(\mathbb{R}^n)$ can be equipped with the inner product

$$(u, v)_{H^s(\mathbb{R}^n)} = (\mathcal{J}^s u, \mathcal{J}^s v)_{L^2(\mathbb{R}^n)}$$

and the induced norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|\mathcal{J}^s u\|_{L^2(\mathbb{R}^n)}$$

It is then immediate that $H^s(\mathbb{R}^n)$ is a seperable Hilbert space.

Lemma 2.1 *$\mathcal{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.*

Proof More precisely, we will show that for a fixed $u \in H^s(\mathbb{R}^n)$ and $\varepsilon > 0$, $\exists\, v \in \mathcal{D}(\mathbb{R}^n)$ such that $\|u - v\|_{H^s(\mathbb{R}^n)} < \varepsilon$. Choose $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi(x) = 1$ for $x \leq 1$ and $\chi(x) = 0$ for $x \geq 2$. For $\delta > 0$, define $\chi_\delta \in \mathcal{D}(\mathbb{R}^n)$ by $\chi_\delta(x) = \chi(\delta x)$. Then for $\psi \in \mathcal{S}(\mathbb{R}^n)$ we have $\chi_\delta \psi \in \mathcal{D}(\mathbb{R}^n)$ and $\chi_\delta \psi \to \psi$ in $\mathcal{S}(\mathbb{R}^n)$ as $\delta \to 0 \Rightarrow \mathcal{J}^s \chi_\delta \psi \to \mathcal{J}^s \psi$ in $\mathcal{S}(\mathbb{R}^n)$ as $\delta \to 0 \Rightarrow \mathcal{J}^s \chi_\delta \psi \to \mathcal{J}^s \psi$ in $L^2(\mathbb{R}^n)$ as $\delta \to 0$.

Since $\mathcal{S}(\mathbb{R}^n)$ in dense in $L^2(\mathbb{R}^n)$, $\exists\, g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\mathcal{J}^s u - g\|_{L^2(\mathbb{R}^n)} \leq \varepsilon/2$ and $u = \mathcal{J}^{-s} g \in \mathcal{S}(\mathbb{R}^n)$. Let $\psi = \mathcal{J}^{-s} g$ and choose δ sufficiently small such that $\|\mathcal{J}^s \chi_\delta \psi - g\|_{L^2(\mathbb{R}^n)} \leq \varepsilon/2$. Then we choose $v = \chi_\delta \psi \Rightarrow \|u - v\|_{L^2(\mathbb{R}^n)} \leq \|u - g\|_{L^2(\mathbb{R}^n)} + \|g - \mathcal{J}^s \chi_\delta \psi\|_{L^2(\mathbb{R}^n)} \leq \varepsilon$. ∎

This immediately implies

Corollary 2.2 *$\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.*

We can also show the following.

Lemma 2.3 *If* $s \leq t$ *then* $H^t(\mathbb{R}^n) \subseteq H^s(\mathbb{R}^n)$ *and* $\|u\|_{H^s(\mathbb{R}^n)} \leq \|u\|_{H^t(\mathbb{R}^n)}$.

Proof If $u \in H^t(\mathbb{R}^n)$ then $\exists\, g \in L^2(\mathbb{R}^n)$ such that

$$\langle \mathcal{J}^t u, \phi \rangle = (g, \phi)_{L^2(\mathbb{R}^n)}, \forall\, \phi \in \mathcal{S}(\mathbb{R}^n)$$

This implies

$$\langle \mathcal{J}^s u, \phi \rangle = (\mathcal{J}^{s-t} g, \phi)_{L^2(\mathbb{R}^n)}$$

and of course

$$\|u\|_{H^t(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} \geq \|\mathcal{J}^{s-t} g\|_{L^2(\mathbb{R}^n)} = \|u\|_{H^s(\mathbb{R}^n)} \qquad \blacksquare$$

We can generalize Sobolev spaces to closed sets $F \subseteq \mathbb{R}^n$.

Definition For any closed set $F \subseteq \mathbb{R}^n$, the *associated Sobolev space of order* s, denoted H^s_F, is defined by

$$H^s_F = \{u \in H^s(\mathbb{R}^n) : \text{supp} u \subseteq F\}$$

Lemma 2.4 H^s_F *is a closed subspace of* $H^s(\mathbb{R}^n)$.

Proof Suppose a sequence $(u_i)_{i=1}^{\infty}$ in H^s_F converges to $u \in H^s(\mathbb{R}^n)$. If $\phi \in \mathcal{D}(F^c)$ then let $\tilde{\phi}$ denote the extension of ϕ to $\mathcal{D}(\mathbb{R}^n)$ by zero. Then we have

$$\langle u|_{F^c}, \phi \rangle = \langle u, \tilde{\phi} \rangle = \langle u - u_i, \tilde{\phi} \rangle + \langle u_i, \tilde{\phi} \rangle = \langle u - u_i, \tilde{\phi} \rangle = (\mathcal{J}^s u - \mathcal{J}^s u_i, \tilde{\phi})_{L^2(\mathbb{R}^n)}$$

and by the Cauchy-Schwarz inequality

$$|\langle u|_{F^c}, \phi \rangle| \leq \|\mathcal{J}^s u - \mathcal{J}^s u_i\|_{L^2(\mathbb{R}^n)} \|\tilde{\phi}\|_{L^2(\mathbb{R}^n)} = \|u - u_i\|_{H^s(\mathbb{R}^n)} \|\tilde{\phi}\|_{L^2(\mathbb{R}^n)}$$

thus we have

$$\langle u|_{F^c}, \phi \rangle = 0, \forall\, \phi \in \mathcal{D}(F^c) \Rightarrow \text{supp} u \subseteq F \qquad \blacksquare$$

Since H^s_F is a closed subspace of $H^s(\mathbb{R}^n)$, it is therefore a Hilbert space when equipped with the restriction of the inner product of $H^s(\mathbb{R}^n)$.

We can now prove the Sobolev imbedding theorem, which states that if s is a sufficiently large positive number then the elements of $H^s(\mathbb{R}^n)$ are equivalent to Hőlder continuous functions.

Theorem 2.5 *Suppose $0 < \mu < 1$. If $u \in H^{n/2+\mu}(\mathbb{R}^n)$, then u has an ae Hőlder-continuous representative in $L^2(\mathbb{R}^n)$. In fact,*

$$|h(x)| \leq C\|u\|_{H^{n/2+\mu}(\mathbb{R}^n)}$$

and

$$|h(x) - h(y)| \leq C'\|u\|_{H^{n/2+\mu}(\mathbb{R}^n)}|x-y|^{\mu}$$

for $x, y \in \mathbb{R}^n$, where h is the $L^2(\mathbb{R}^n)$ representative of u.

Proof If $u \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \exists\, g \in \mathcal{S}(\mathbb{R}^n)$ such that $\forall\, \phi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\langle \mathcal{J}^{n/2+\mu}u, \phi\rangle = (g, \phi)_{L^2(\mathbb{R}^n)} \Rightarrow \langle u, \phi\rangle = (\mathcal{J}^{-n/2-\mu}g, \phi)_{L^2(\mathbb{R}^n)}$$

Let $h = \mathcal{J}^{-n/2-\mu}g \in \mathcal{S}(\mathbb{R}^n)$. Now by the Fourier inversion formula

$$|h(x)| = \left|\int_{\mathbb{R}^n} \hat{h}(\xi)e^{2\pi i\xi\cdot x}d\xi\right| \leq \int_{\mathbb{R}^n} |\hat{h}(\xi)|d\xi$$

and then by the Cauchy-Schwarz inequality and Plancherel's theorem

$$\begin{aligned}
|h(x)| &\leq \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{-\frac{n}{2}-\mu}{2}}|(1+|\xi|^2)^{\frac{\frac{n}{2}+\mu}{2}}\hat{h}(\xi)|d\xi \\
&\leq \left\|(1+|\cdot|^2)^{\frac{-\frac{n}{2}-\mu}{2}}\right\|_{L^2(\mathbb{R}^n)} \cdot \left\|(1+|\cdot|^2)^{\frac{\frac{n}{2}+\mu}{2}}\hat{h}(\cdot)\right\|_{L^2(\mathbb{R}^n)} \\
&= C\|\mathcal{J}^{\frac{n}{2}+\mu}h\|_{L^2(\mathbb{R}^n)} = C\|u\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)}
\end{aligned}$$

Now take $u \in H^{\frac{n}{2}+\mu}(\mathbb{R}^n)$. From Lemma 2.1 we know $\exists$ a sequence (u_i) in $\mathcal{D}(\mathbb{R}^n)$ such that $u_i \to u$ in $H^{\frac{n}{2}+\mu}(\mathbb{R}^n)$. Now

$$|h_j(x) - h_k(x)| \leq C\|u_j - u_k\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)}$$

which implies (h_j) is a uniformly Cauchy sequence of $\mathcal{D}(\mathbb{R}^n)$ functions. Thus

$$H(x) = \lim_{j\to\infty} h_j(x)$$

is a continuous function and $h_j \to H$ uniformly. In fact, H is uniformly continuous since

$$|H(x) - H(y)| \leq |H(x) - h_j(x)| + |h_j(x) - h_j(y)| + |H(y) - h_j(y)|$$

and each h_j is uniformly continuous. Also, because $h_j \to H$ uniformly and each $h_j \in L^2(\mathbb{R}^n)$, we have $H \in L^2(\mathbb{R}^n)$.

For $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$(h_j, \phi)_{L^2(\mathbb{R}^n)} \to (h, \phi)_{L^2(\mathbb{R}^n)} \text{ and } (h_j, \phi)_{L^2(\mathbb{R}^n)} \to (H, \phi)_{L^2(\mathbb{R}^n)}$$
$$\Rightarrow (h, \phi)_{L^2(\mathbb{R}^n)} = (H, \phi)_{L^2(\mathbb{R}^n)} \Rightarrow h = H \text{ for ae } x \in \mathbb{R}^n$$

Thus for ae x

$$|h(x)| = |H(x)| = \lim_{j\to\infty} |h_j(x)| \le C \lim_{j\to\infty} \|u_j\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)} = C\|u\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)}$$

and the first inequality is proved.

Similarly, for $u \in H^{\frac{n}{2}+\mu}(\mathbb{R}^n)$, define $\delta_t h(x) = h(x+t) - h(x)$. Then by the Fourier inversion formula we have

$$|\delta_t h(x)| \le \int_{\mathbb{R}^n} |e^{2\pi i t\cdot\xi} - 1||\hat{h}(\xi)|d\xi = \int_{\mathbb{R}^n} |e^{2\pi i t\cdot\xi} - 1|(1+|\xi|^2)^{\frac{-\frac{n}{2}-\mu}{2}}|(1+|\xi|^2)^{\frac{\frac{n}{2}+\mu}{2}}\hat{h}(\xi)|d\xi$$

and by the Cauchy-Schwarz inequality

$$|\delta_t h(x)| \le M_\mu(t)\|u\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)}$$

where

$$(M_\mu(t))^2 = \int_{\mathbb{R}^n} |e^{2\pi i t\cdot\xi} - 1|^2(1+|\xi|^2)^{-\frac{n}{2}-\mu}d\xi$$

Note $|e^{2\pi i t\cdot\xi} - 1|^2 = 2(1 - \cos(2\pi t\cdot\xi))$ and so $\exists$ constant D such that for $0 < |\xi\cdot t| \le 1$ we have $|e^{2\pi i t\cdot\xi} - 1| < D|\xi\cdot t|$, which implies

$$\begin{aligned}
(M_\mu(t))^2 &\le \int_{|\xi|<1/|t|} |e^{2\pi i t\cdot\xi} - 1|^2(1+|\xi|^2)^{-\frac{n}{2}-\mu}d\xi + \int_{|\xi|\ge 1/|t|} |e^{2\pi i t\cdot\xi} - 1|^2(1+|\xi|^2)^{-\frac{n}{2}-\mu}d\xi \\
&\le D\int_{|\xi|<1/|t|} |\xi\cdot t|^2(1+|\xi|^2)^{-\frac{n}{2}-\mu}d\xi + 4\int_{|\xi|\ge 1/|t|} (1+|\xi|^2)^{-\frac{n}{2}-\mu}d\xi \\
&\le D|t|^2\int_{|\xi|<1/|t|} |\xi|^2(1+|\xi|^2)^{-\frac{n}{2}-\mu}d\xi + 4\int_{|\xi|\ge 1/|t|} (1+|\xi|^2)^{-\frac{n}{2}-\mu}d\xi \\
&\le D|t|^2\int_{|\xi|<1/|t|} |\xi|^2(|\xi|^2)^{-\frac{n}{2}-\mu}d\xi + 4\int_{|\xi|\ge 1/|t|} (|\xi|^2)^{-\frac{n}{2}-\mu}d\xi \\
&= D|t|^2\int_0^{1/|t|} \rho^2\rho^{-n+2\mu}\rho^n d\rho + 4\int_{1/|t|}^\infty \rho^{-n-2\mu}\rho^n d\rho \quad \text{in radial coordinates} \\
&= D|t|^2|t|^{2\mu-2} + 4|t|^{2\mu} \le E|t|^{2\mu} \quad \text{where E is a constant}
\end{aligned}$$

Thus we have

$$|\delta_t h(x)| \leq C'|t|^\mu \|u\|_{H^{\frac{n}{2}+\mu}(\mathbb{R}^n)} \quad \forall\, x, t \in \mathbb{R}^n$$

which proves the second inequality. ∎

References

[1] McLean, William. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge: Cambridge University Press, 2000.