Poisson Equation on Closed Manifolds

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1 Introduction

The purpose of this project is to investigate the poisson equation $\Delta\phi = \rho$ on closed manifolds (compact manifolds without boundary), where ρ is known, and ϕ is unknown. A particular emphasis will be placed on Riemannian manifolds.

We begin by introducing the basic machinery of differential geometry that will be needed. For example, we will define smooth manifolds, Riemannian manifolds, metric tensors, and other basic differential geometry constructs. We will investigate how to properly generalize the Laplacian Δ on $\mathbb{R}^n$ to a Riemannian manifold. To do this we will need to define things such as the divergence of a vector field on a manifold, the (metric induced) covariant derivative, differential forms, the Hodge star operator, and the space of L^2 functions on a manifold.

We then have the necessary machinery to prove the first of two main results considered in this project. We prove the existence and uniqueness of a smooth solution to the poisson equation $\Delta\phi = \rho$ for smooth ρ. We do this by using Riesz representation theorem to prove the existence of a weak solution, and then we show that this weak solution is in fact smooth.

We will conclude by proving a Schauder estimate for elliptic operators on complete, compact manifolds.

2 Geometric background

We begin by briefly reminding the reader of the definition of a manifold and a smooth manifold [1].

Definition 1. *A manifold M of dimension n is a topological space that is Hausdorff, second countable, and every point of M has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$.*

A manifold as defined above is an adequate setting for investigating topological properties, but in order to perform analysis on a manifold, we must introduce more (differentiable) structure.

If M is a manifold of dimension n, a *coordinate chart* (often just referred to as a *chart*) on M is a pair (U, φ) where U is an open subset of M and $\phi : U \to \tilde{U}$ is a homeomorphism from U to an open subset $\tilde{U} = \varphi(U) \subset \mathbb{R}^n$. Clearly each point $x \in M$ is contained in at least one chart.

If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the transition map from φ to ψ. (V, ψ) and (U, φ) are said to be smoothly compatible if either $U \cap V = \emptyset$ or the transition map just defined is a diffeomorphism.

An *atlas* for M is a collection of coordinate charts whose domain covers M. An atlas is smooth if all charts are smoothly compatible. A manifold with a maximal smooth atlas is said to be a smooth manifold. If all of the transition maps are merely differentiable, M is said to be a differentiable manifold. If all transition maps are k times continuously differentiable then M is said to be a C^k manifold.

For a given smooth manifold, there is no natural generalization of the Laplacian on $\mathbb{R}^n$ (or indeed the well known Laplacian on the n-sphere) without additional geometric structure on the manifold in the form a Riemannian metric [2]. Hence we will focus on *Riemannian manifolds*, defined as follows:

Definition 2. *A* Riemannian manifold (M, g) *is a smooth manifold M with a family of varying positive definite inner products $g = g_x$ on T_xM for every $x \in M$. The family g is called the* Riemannian metric*.*

For the remainder of this text, when we say manifold we will be referring to a Riemannian manifold, and rather than writing (M, g) we will simply write M (unless the context warrants that the metric g be specified).

On $\mathbb{R}^n$ a natural choice of Riemannian metric is given by the usual inner product from vector calculus, $g_x(u, v) = u \cdot v$ for all $u, v \in T_x\mathbb{R}^n$, for all $x \in \mathbb{R}^n$. $\mathbb{R}^n$ endowed with this metric is often called *Euclidean space*.

An interesting theorem of differential geometry, the *Whitney embedding theorem*, states that every manifold possesses a Riemannian metric.

To do computations with a Riemannian metric, it is often necessary to work in a local coordinate chart, and we represent $g_{ij}(x)$ in this chart as

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j.$$

At this point I should mention that I will, as is conventional in differential geometry, utilize the *Einstein summation convention*, where if an index appears more than once in the same term, then we are summing over that index.

An interesting side note that can relate geometry with elementary analysis familiar to many is that Riemannian manifolds are also metric spaces. If

$\gamma : [a, b] \to M$ is a C^1 curve on M then we define its length $L(\gamma)$ by

$$L(\gamma) = \int_a^b ||\gamma'(t)||dt,$$

where $||\cdot||$ is the norm induced by the metric, given by $||\gamma'(t)|| = \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}$. The *Riemannian distance* between points p and q on M is the infinum of the length of all piecewise smooth curves joining p and q. (M, d) is a metric space [3].

A very important definition when working on manifolds is the following (recalling that a one-form is a linear function from $T_x(M) \to \mathbb{R}$) (see, for example, [4] for a more complete definition):

Definition 3. *A* differential k-form *is a tensor of rank k that is antisymmetric under exchange of any pair of indices. It is possible to write a p-form α in coordinates by*

$$\alpha = \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

3 L^2 spaces of functions

At this point we have the necessary machinery in place to develop the necessary analysis on our Riemannian manifold. To avoid some problems later on, in this section we will assume that M is oriented and connected (recall that, roughly speaking, orientable in the manifold setting means that one can choose a "clockwise" orientation for every loop on the manifold. For a surface embedded in $\mathbb{R}^n$ this amounts to being able to make a consistent choice of unit normal).

We wish to construct a Hilbert space of real values functions on M. To do this we seek an n form $\alpha(x)$ such that $\langle f, g\rangle := \langle f, g\rangle_M := \int_M f(x)g(x)\alpha(x)$ defines a positive definite inner product space (recalling that n forms are the objects that transform correctly to give an integral on a manifold). An α requiring this property is called a *volume form.* The following choice of volume form is derived in [2]:

Definition 4. *The* volume form *of a Riemannian metric is the form dvol, given in local coordinates by*

$$\text{dvol} = \sqrt{\det g} dx^1 \wedge \cdots \wedge dx^n,$$

where $(\partial_{x^1}, \ldots, \partial_{x^n})$ *is a positively oriented basis of* T_xM*. The* volume *of* (M, g) *is defined to be*

$$\mathrm{vol}(M) = \int_M \mathrm{dvol}(x).$$

Definition 5. *We define the Hilbert space* $L^2(M, g)$ *to be the completion of* $C_0^\infty(M)$ *with respect to the inner product* $\langle f, g\rangle = \int_M f(x)g(x)\,\mathrm{dvol}$.

We can now set $L^2\Lambda^1T^*(M, g)$ to be the completion of $C_0^\infty T^*M$ with respect to the global inner product

$$\langle \omega, \eta\rangle = \int_M g(\omega, \eta)\,\mathrm{dvol}(x).$$

In a similar fashion, g induces an inner product g on each product $T_xM \otimes \cdots \otimes T_xM$, and so on each $\Lambda^k T_x^*M$, and so a global inner product is given by

$$\langle \alpha, \beta\rangle = \int_M g(\alpha, \beta)\,\mathrm{dvol}, \quad \alpha, \beta \in C_0^\infty \Lambda^k T^*M.$$

The completion is denoted $L^2\Lambda^kT^*M$.

4 The Laplacian defined for functions on a manifold

We now have at our disposal the geometric constructs necessary to define the Laplacian for functions on a manifold. When defining such an operator, we obviously want the Laplacian to agree with (up to a sign) the Laplacian $\delta^{ij}\partial_i\partial_j$ on $\mathbb{R}^n$ (here δ is the Kronecker delta). Unlike $\mathbb{R}^n$ however, we do not, in general, have globally defined coordinates and so we need a coordinate free expression in our generalization. Recall from vector calculus in $\mathbb{R}^n$ that the Laplacian may be written as

$$\Delta f = \delta_i{}^j\partial_i\partial_j f = (\mathrm{div} \circ \nabla)f.$$

The operator $\mathrm{div} \circ \nabla$ generalizes naturally to Riemannian manifolds. Because we have a Riemannian metric, we have, via the Levi-Civita connection [4], a corvariant derivative operator ∇. Acting on functions in local coordinates, one can think of this as the directional derivative.

To define div, we observe that, using integration by parts, for $f \in C_0^\infty(\mathbb{R}^n)$, gives the identitiy:

$$-\int_{\mathbb{R}^n} (\partial_i X^i) f = \int_{\mathbb{R}^n} (\partial_i f) X^i$$

for functions X^i. So in $\mathbb{R}^n$ the divergence $\partial_i X^i$ of a vector field $X = X^i \partial_i$ can be characterized by

$$\langle -\operatorname{div} X, f\rangle = \langle X, \nabla f\rangle, \tag{1}$$

where the inner product above is the usual dot product in $\mathbb{R}^n$. In other words, in $\mathbb{R}^n$, $-\operatorname{div}$ is the adjoint to ∇. It is therefore natural to try and use equation 1 to try and define div since we already have a generalization ∇f and an inner product on Riemannian manifolds.

We now determine the form of such an operator in local coordinates $(x^1, \dots, x^n)$. Let $f \in C_0^\infty(U)$, where U is a coordinate chart image in $\mathbb{R}^n$, and X be a vector field $X = X^i \partial_i \in TM$. Then

$$\begin{aligned}
\langle X, \nabla f\rangle &= \int_M \langle X, \nabla f\rangle \operatorname{dvol} = \int_U \langle X^i \partial_i, g^{jk} \partial_j f \partial_j\rangle \operatorname{dvol} \\
&= \int_U X^i (\partial_k f) g^{jk} g_{ij} \sqrt{\det g}\, dx^1 \dots dx^n \\
&= \int_U X^i (\partial_i f) \sqrt{\det g}\, dx^1 \dots dx^n \\
&= -\int_U \frac{1}{\sqrt{\det g}} f \cdot \partial_i (X^i \sqrt{\det g}) \sqrt{\det g}\, dx^1 \dots dx^n \\
&= \left\langle f, -\frac{1}{\sqrt{\det g}} \partial_i (X^i \sqrt{\det g}) \right\rangle
\end{aligned}$$

So the div of X will satisfy

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \partial_i (X^i \sqrt{\det g}).$$

This expression is independent of choice of coordinates and so we define the Laplacian on functions to be $\Delta := -\operatorname{div} \circ \nabla$. This definition of the Laplacian is sometimes called the *Laplace-Beltrami operator*. In local coordinates:

$$\begin{aligned}
\Delta f &= -\frac{1}{\sqrt{\det g}} \partial_j (g^{ij} \sqrt{\det g} \partial_i f) \\
&= -g^{ij} \partial_j \partial_i f + \text{ (lower order)}.
\end{aligned}$$

It is clear that in $\mathbb{R}^n$ this reduces to $-\delta_i{}^j\partial_j\partial_i f$, the usual Laplacian.

By writing (1) in terms of differential forms, we gain another useful characterization of the Laplacian. For a one form α, we have

$$g(\alpha(X), df) = g(X, \nabla F)$$

for any vector f and any function f. Let $\delta : \Lambda^1 T^*M \to C^\infty(M)$ be defined by $\delta(\omega) = -\operatorname{div}(\alpha^-1(\omega))$ where α is the bundle isomorphism $TM \to T^*M$. Then δ is characterized by

$$\langle \delta\omega, f\rangle = \langle \omega, df\rangle$$

Now as a second, coordinate free definition of Δ, we write $\Delta = \delta d$, where d is the exterior derivative. We may also write this in terms of the *Hodge star operator*, a pointwise isometry $* = *_x : \Lambda^k T_x^*M \to \Lambda^{n-k} T_x^*M$. Choosing a positively oriented orthonormal basis $\{\theta^1, \ldots, \theta^n\}$ of T_x^*M, we define $*$ by its action on basis elements $\theta^{i_1} \wedge \ldots \wedge \theta^{i_k}$ of $\lambda_x^k T^*M$, given by

$$*(\theta^{i_1} \wedge \ldots \theta^{i_k}) = \theta^{j_1} \wedge \ldots \wedge \theta^{j_{n-k}}$$

where $\theta^{i_1} \wedge \ldots \wedge \theta^{i_k} \wedge \theta^{j_1} \wedge \ldots \wedge \theta^{j_{n-k}} = \mathrm{dvol}(x)$.

By somewhat lengthy but straightforward calculations it can be shown that [2]

$$\Delta f = \delta df = - * d * df.$$

This operator is known as the *Hodge Laplacian*. It agrees with the Laplace-Beltrami operator on functions, but differs in general on differential forms.

5 The Poisson equation on a manifold

We are now ready to consider the equation $\Delta\phi = \rho$, where ρ is given and ϕ is to be found, defined on a closed (compact without boundary) Riemannian manifold.

We recall, see for example [6], that on a closed manifold we may integrate according to the following rule:

$$\int_M u\Delta v - v\Delta u = \int_M v\Delta u.$$

Hence setting $u = 1$, $v = \phi$ we get

$$\int_M \Delta\phi = 0.$$

So we conclude that a necessary criterion to solve our problem is that

$$\int_M \rho = 0.$$

Throughout this section we will work towards proving the following theorem, following the steps of [5]:

Theorem 1. *On a closed Riemannian manifold M, if ρ is a smooth function of integral 0 then there is a smooth solution of the equation $\Delta\phi = \rho$, unique up to the addition of a constant.*

A C^∞ function ϕ satisfies $\Delta\phi = \rho$ if and only if for all test functions we have

$$\langle \nabla f, \nabla \phi \rangle = \langle f, \rho \rangle.$$

Define the following norm on the space of smooth function of integral zero C_0^∞ by

$$||f||_H^2 = \int_M |\nabla f|^2.$$

There is an associated inner product with this norm [7] (to see this note that this norm is in fact equivalent to the usual norm given to the Sobolev space H^1), so C_0^∞ is a pre-Hilbert space, and we denote its completion by H. By the Reisz representation theorem of functional analysis, for any bounded linear map $\alpha : H \to \mathbb{R}$ there is a unique $a \in H$ such that $\alpha(f) = \langle a, f \rangle_H$ for all $f \in H$. Then $\alpha_\rho(f) = \langle \rho, f \rangle$ is a linear map from C_0^∞ to $\mathbb{R}$.

Suppose for the moment that this extends to a bounded map on H. Then by the representation theorem there exists $\phi \in H$ such that

$$\langle \nabla\phi, \nabla f \rangle = \langle \rho, f \rangle,$$

for all f. Then ϕ is known as a weak solution to the poisson equation. Thus we can prove the theorem by proving the following two assertions:

1. α_p extends to a bounded linear map on H.

2. A weak solution $f \in H$ is actually smooth.

We observe that the first assertion follows from the *Poincare inequality* for Riemannian manifolds (for a proof of this inequality see [8]), namely that

$$\int_M |\phi|^2 \leq C^2 \int_M |\nabla \phi|^2$$

for all ϕ of integral zero.

We now seek to prove the smoothness of a weak solution. Suppose that $\phi \in H$ is a weak solution. Let $\{\phi_i\}$ be a Cauchy sequence of smooth functions of integral 0 and for any ψ we have

$$\langle \phi_i, \psi \rangle_H \to \alpha_\rho(\psi),$$

as $i \to \infty$. Appealing once more to Poincare's inequality, the sequence is Cauchy in L^2 and so has a limit ϕ in L^2. We need to show that ϕ is smooth. Smoothness is a local property, so we will restrict our analysis to a single chart, and so we prove it for a bounded open set $\Omega \subset \mathbb{R}^n$:

Theorem 2. *Let Ω be a bounded open set in $\mathbb{R}^n$ and ρ be a smooth function on Ω. Suppose ϕ is an L^2 function on Ω with the property that for any smooth function χ of compact support in Ω,*

$$\int_\Omega \Delta \chi \phi = \int_\Omega \chi \rho.$$

Then ϕ is smooth and satisfies the equation $\Delta \phi = \rho$.

Proof. We first reduce to the case where ρ vanishes everywhere. Smoothness is local, so it suffices to prove the claim that ϕ is smooth over an arbitrary set $\Omega' \subset \Omega$. Now we choose ρ' equal to ρ on a neighbourhood of the closure of Ω' and of compact support in Ω. Then if we find a smooth solution ϕ' of $\Delta \phi' = \rho'$ over Ω then $\psi = \phi - \phi'$ will be a weak solution of $\Delta \psi = 0$ on Ω'. Then if we show that ψ is smooth so will be ϕ. But we can find a solution using the Newtonian potential; if τ has compact support in Ω and equal to 1 on Ω' then $K * (\tau \rho)$ satisfies $\Delta \phi' = \rho$ over Ω'.

So we have reduced the problem to the case $\rho = 0$. We suppose that ϕ is a weak solution of $\Delta \phi = 0$ on Ω and seek to prove that ϕ is smooth in the interior domian Ω'. Fix a smooth function β on $\mathbb{R}$ with $\beta(r)$ constant for small r and vanishing for $r \geq \epsilon$, with the property that

$$|S^{n-1}| \int_0^\infty r^{n-1} \beta(r) dr = 1.$$

Now let $B(z) = \beta(|z|)$ on $\mathbb{R}^n$. B as defined is smooth and has integral 1 over $\mathbb{R}^n$. If ψ is a smooth harmonic function on a neighbourhood of the closed ϵ-ball centered at the origin then (using mean value property and switching to polar coordinates):

$$\int_{\mathbb{R}^n} B(-z)\psi(z) = \int_0^\infty \int_{S^{n-1}} r^{n-1}\beta(r)\psi(r,\theta)d\theta dr = \psi(0).$$

But this integral is just the convolution $B * \psi$ at 0. Hence by translation invariance we have shown that for any smooth ψ on $\mathbb{R}^n$ with $\Delta\psi$ supported in a compact set $J \subset \mathbb{R}^n$ that $B*\psi - \psi$ vanishes outside the ϵ neighbourhood of J. In particular, if ϕ is smooth on Ω then $B*\phi = \phi$ in Ω'. We also known (elementary analysis) that for any L^2 function ϕ that $B*\phi$ is smooth. Hence proving the smoothness of ϕ in Ω' is equivalent to establishing that $B*\phi = \phi$ in Ω'.

To do this it is sufficient to show that for any test function χ in Ω' that

$$\langle \chi, \phi - B*\phi\rangle_{L^2} = 0,$$

where we have the L^2 inner product

$$\langle f, g\rangle_{L^2} = \int fg.$$

Let $h = K*(\chi - B*\chi) = K*\chi - B*K*\chi$. $K*\chi$ is smooth on $\mathbb{R}^n$ and since K is the Newtonian kernel we have that $\Delta(K*\chi) = \chi$. This impies that $\Delta(K*\chi)$ vanishes outside the support of χ and by what we showed above $B*K*\chi = K*\chi$ outside the ϵ-neighbourhood of the support of χ. So h is of compact support contained in Ω and h can be used as a test function in the hypothesis that $\Delta\phi = 0$ in the weak sense, i.e. $\langle \Delta h, \phi\rangle = 0$. But $\Delta h = \Delta(K*(\chi - B*\chi)) = \chi - B*\chi$, hence

$$\langle \chi - B*\chi, \phi\rangle = 0$$

and applying the identity proved above yields

$$\langle \chi, \phi - B*\phi\rangle = 0$$

as was to be shown. □

Hence we have shown that

1. α_p extends to a bounded linear map on H.

2. A weak solution $f \in H$ is actually smooth.

and so have in fact proven theorem 1.

6 A Schauder estimate on complete closed manifolds

To conclude this project we will go over a proof of a Schauder estimate for elliptic operators on closed manifolds. In fact, the analysis to follow is valid not only for closed manifolds but more generally for any compact manifold. The proof to follow is based on the proof given in [9].

Let M be a compact Riemannian manifold of dimension n. We now generalize various definitions to the case of a compact Riemannian manifold:

Definition 6. *The C^k* norm corresponding to the metric g of M *is given by*

$$|u|_{C^k(M)} = |u|_{0,M} + \sum_{l=1}^{k} \sup_{x \in M} ||\nabla^l u(x)||_g,$$

where $|\cdot|_{0,M}$ is the supremum norm on M and $||\nabla^l u||_g$ is the norm induced by g of the tensor $\nabla^l u$.

Recall that Riemannian manifolds are metric spaces, with the distance $d(x,y)$ between two points $x, y \in M$ defined in the geometric background section. We define the Hölder semi-norm for functions $[u]_{\alpha,M}$ of exponent α as

$$[u]_{\alpha,M} := \sup_{x,y \in M} \frac{|u(x) - u(y)|}{(d(x,y)^\alpha)}, \quad u \in C^0(M).$$

We now define a Hölder semi-norm for tensors. This is a little bit more complicated however since we cannot write, (for a tensor T and points $x, y, x \neq y$) $T(x) - T(y)$. To generalize the Hölder semi-norm to tensors, we make use of parallel transport, i.e. let $\tau_{(x,y)} : T_x M \to T_y M$ be the isomorphism of tangent spaces along the geodesic connecting x to y. We now define the Hölder semi-norm for a tensor T as

$$[T]_{\alpha,M} := \sup_{x,y \in M} \frac{||T(x) - \tau^*_{(x,y)}(T(y))||_g}{(d(x,y)^\alpha)}.$$

We define the kth order Hölder coefficient as

$$[u]_{k,\alpha,M} := [\nabla^k u]_{\alpha,M}$$

and the $C^{k,\alpha}$ Hölder norm of $u \in C^k(M)$ as

$$|u|_{C^{k,\alpha}(M)} := |u|_{C^k(M)} + [u]_{k,\alpha,M}.$$

We call the space $C^{k,\alpha}(M)$ the space of C^k functions on M with finite $C^{k,\alpha}$ norm.

We now consider the case where $\Omega \subseteq M$ is a domain. We can then consider the Hölder norms on Ω by taking all suprema in the definitions above to be over Ω and not M. In a sufficiently small domain Ω there exist so called *Riemannian normal coordinates* in which $g_{ij}(x) = \delta_{ij}$ and $g_{ij,k} = 0$ at a point x in the center of the coordinate chart. By continuity, for any $a \in \mathbb{R}$ there exists a number $r(x)$ such that if Ω is a domain containing x of diameter less than $r(x)$ then

$$|g_{ij} - \delta_{ij}|_0 + |g_{ij,k}|_0 \leq a, \quad \text{in } \Omega.$$

Furthermore, M is compact so there is a minimal diameter r_g. We will call r_g the radius of uniformity of the metric g.

Proposition 1. *Let Ω be any domain of diameter less than r_g. Then there exists a constant C (depending only on n) such that*

$$\frac{1}{C}|u|^{\delta}_{k,\alpha,\Omega} \leq |u|_{C^{k,\alpha}(\Omega)} \leq C|u|^{\delta}_{k,\alpha,\Omega},$$

for any $u \in C^{k,\alpha}(M)$. $|\cdot|^{\delta}_{k,\alpha,\Omega}$ refers to the $C^{k,\alpha}$ with respect to the norm on Ω induced by the Euclidean metric when Ω is viewed as a subset of R^n.

Proof. Choosing Riemannian normal coordinates, we may estimate the supremum norm in a straightforward fashion by expression the covariant derivatives in terms of ordinary derivatives and the connection coefficients. Note that the Hölder coefficient bounds involve parallel transport, which can be expressed as the solution of an ODE with initial conditions. Recall that the ODE is given by

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0.$$

Hence we can bound the parallel transport in terms of the initial conditions, giving estimates on the Hölder coefficient. □

We now suppose that L is an elliptic operator $L = a^{ij}\nabla_i\nabla_j + b^i\nabla_i + c$ where a symmetric and positive definite tensor, b is a $C^{0,\alpha}$ vector field on M and $c \in C^{0,\alpha}(M)$, and that L satisfies the conditions

$$||a||_{C^{0,\alpha}(M)} + ||b||_{C^{0,\alpha}(M)} + ||c||_{C^{0,\alpha}(M)} \leq \Lambda,$$

$$\lambda||\xi||^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda||\xi||^2, \quad \text{for all } x \in M, \xi \in \mathbb{R}^n.$$

Consider the following problem:

$$Lu = f \text{ in } M,$$

if M is closed and

$$\begin{aligned} Lu &= f \text{ in } M \\ u &= g \text{ on } \partial M \end{aligned}$$

if M is open.

We can now prove the following Schauder estimate:

Theorem 3. *Let L be as above. Then there exists a constant C depending on M, λ, Λ such that*

$$|u|_{C^{2,\alpha}}(M) \leq C(|Lu|_{C^{0,\alpha}} + |u|_{0,M}),$$

valid for all $u \in C^{2,\alpha}(M)$.

Proof. Let (U_i, ϕ_i) be a covering of M by a finite number (possible because our manifold is compact) of normal coordinate charts with diameter less than r_g.

$$\begin{aligned} |u|_{C^2(M)} &\leq \sum_i |u|_{C^2(U_i)} \\ &\leq C \sum_i |u \circ \phi_i^{-1}|^{\delta}_{C^2(\phi_i(U_i))} \\ &\leq C \sum_i |u \circ \phi_i^{-1}|^{\delta}_{k,\alpha,\phi_i}(U_i) \\ &\leq C \sum_i (|\phi_i^* L(u \circ \phi_i^{-1})|^{\delta}_{0,\alpha,\phi_i(U)} + |u \circ \phi_i^{-1}|_{0,\phi_i(U_i)}), \end{aligned}$$

$\phi_i^* L$ denoting the operator L written in local coordinates. Applying the metric bounds

$$\begin{aligned} |u|_{C^2(M)} &\leq C \sum_i (|Lu|_{C^{0,\alpha}(U_i)} + |u|_{0,U_i}) \\ &\leq C(|Lu|_{C^{0,\alpha}(M)} + |u|_{0,M}). \end{aligned}$$

□

We now estimate the Hölder coefficient in the two possible cases: if x and y are points in M with $d(x,y)\langle r_g$ and $d(x,y) \geq r_g$. In case one consider a normal coordinate neighbourhood (Ω, ϕ) containing x and y. Then

$$
\begin{aligned}
\frac{||\nabla^2 u(x) - \tau^*_{(x,y)} \nabla^2 u(y)||^g}{(d(x,y)^\alpha)} &\leq [\nabla^2 u]^g_{\alpha,\Omega} \\
&\leq C[\phi^*(\nabla^2 u)]^\delta_{\alpha,\phi(\Omega)} \text{ using metric bounds} \\
&\leq C(|\phi^* L(u \circ \phi)|^\delta_{0,\alpha,\phi(\Omega)} + |u \circ \phi|_{0,\phi(\Omega)}) \\
&\leq C(|Lu|_{C^{0,\alpha}(M)} + |u|_{0,M}).
\end{aligned}
$$

In the case $d(x,y) \geq r_g$

$$
\begin{aligned}
\frac{||\nabla^2 u(x) - \tau^*_{(x,y)} \nabla^2 u(y)||^g}{(d(x,y))^\alpha} &\leq C|\nabla^2 u|_{0,M} r_g^\alpha \\
&\leq C|u|_{C^k(M)} \\
&\leq C(|Lu|_{C^{0,\alpha}(M)} + |u|_{0,M})
\end{aligned}
$$

Now taking the sup of the quotient above and combining this with the ordinary derivative estimates yields the result.

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