McGill University

MATH 580 Final Project

A Survey on the Methods of Proofs of the Malgrange-Ehrenpreis Theorem

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1 Introduction

In this project we explore the existence of fundamental solutions of linear partial differential operators with constant coefficients. We focus on understanding of various proofs of the Malgrange-Ehrenpreis theorem, which was a first striking evidence of the impact of distribution theory on the study of linear partial differential equations [17].

1.1 Notations and background

In the theory of distributions, a typical space of test functions consists of all smooth (complex-valued) functions on $\mathbb{R}^n$ with compact support. The notations $C_c^\infty(\mathbb{R}^n)$, $C_0^\infty(\mathbb{R}^n)$ and L. Schwartz's [14] original $\mathscr{D}(\mathbb{R}^n)$ are used [4]. We write $\mathscr{D}'(\mathbb{R}^n)$ for the space of distributions.

By using a larger space of test functions, one can define the temperate distributions, a subspace of $\mathcal{D}'(\mathbb{R}^n)$. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the function space of all infinitely differentiable functions that are rapidly decreasing at infinity along with all partial derivatives. A *temperate* distribution is a special case of a distribution, which is a continuous linear form on $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ endowed with a particular topology [12]. These distributions are useful in studies of the Fourier transform in generality: all temperate distributions have a Fourier transform. The space of temperate distributions is defined as the dual of the Schwartz space and is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Definition 1.1. If $T \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform $\mathcal{F}T$ of T is defined by:

$$\mathcal{F}T(\phi) = T(\mathcal{F}\phi) \tag{1.1}$$

for every $\phi \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, we define:

$$\partial^\alpha T(\phi) = (-1)^{|\alpha|} T(\partial^\alpha \phi) \tag{1.2}$$

for every $\phi \in \mathcal{S}(\mathbb{R}^n)$ [12].

Let $P(\xi)$ be a polynomial in n variables with complex coefficients of precise degree $m \geq 0$. The linear differential operator $P(\partial)$ is obtained by replacing ξ_j by $\partial_j = \partial/\partial x_j$. Alternatively, $P(\partial)$ can be written as:

$$P(\partial) := \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha, \tag{1.3}$$

where $\alpha \in \mathbb{N}^n$ is a multi-index [18, p. 182], [8]. We set $P_m(\partial) := \sum_{|\alpha|=m} c_\alpha \partial^\alpha$ for the homogeneous component of P of degree m and write $\deg P$ for the degree of $P(\partial)$, i.e. $\max\{m \in \mathbb{N} : P_m \not\equiv 0\}$. We denote the adjoint operator $P^*(\partial)$ by:

$$P^*(\partial) := \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \bar{c}_\alpha \partial^\alpha \tag{1.4}$$

Definition 1.2. A distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ is called a fundamental solution of $P(\partial) \in \mathbb{C}[\partial_1, \ldots, \partial_n]$ if and only if the equation $P(\partial)E = \delta$ holds in $\mathscr{D}'(\mathbb{R}^n)$.

The importance of the notion of the fundamental solutions is due to the fact that $u = E * f$, where $f \in \mathscr{D}(\mathbb{R}^n)$, gives a solution to the equation $P(\partial)u = f$ [18, p. 182].

We note that the zero differential operator has no fundamental solution, whereas a non-zero identically constant differential operator $P(\partial) \equiv c \in \mathbb{C}[\partial_1, \ldots, \partial_n] \setminus \{0\}$ has a unique fundamental solution $E = \delta/c$. Thus, in general, we do not expect a fundamental solution to be a locally integrable function [11]. We now state the theorem.

Theorem 1.1 (Malgrange [9], Ehrenpreis [3])**.** *Every non-zero linear partial differential operator with constant coefficients has a fundamental solution in the space of distributions, i.e.:*

$$\forall\, P(\partial) \in \mathbb{C}[\partial_1, \ldots \partial_n] \setminus \{0\} : \exists\, E \in \mathscr{D}'(\mathbb{R}^n) : P(\partial)E = \delta \tag{1.5}$$

An immediate corollary of the Malgrange-Ehrenpreis theorem is that every linear differential operator with constant coefficients is locally solvable, and one can deduce regularity properties of the solutions by the examination of the fundamental solution.

It is curious to note that before 1950, when the first edition of [14] appeared, the question about the existence of a fundamental solution was not even raised, since there did not exist a generally accepted definition of a fundamental solution. For instance, before that time, both functions $E = -1/4\pi|x|$ and $F = 1/|x|$ served as fundamental solutions for the three-dimensional Laplacian. Schwartz's definition (cf. Definition 1.2) excludes F, since $\Delta F = -4\pi\delta$

Although the Malgrange-Ehrenpreis theorem states that a non-zero operator $P(\partial) \in \mathbb{C}[\partial_1, \ldots, \partial_n]$ always has at least one fundamental solution, this solution cannot be unique if $\deg P > 0$. In particular, if E is a fundamental solution of $P(\partial)$ and if u is a solution of the homogeneous equation $P(\partial)u = 0$, by the Fundamental theorem of algebra for $\deg P$, $E + u$ is also a fundamental solution of $P(\partial)$ [11].

2 Survey of Existing Proofs of the Malgrange-Ehrenpreis Theorem

In this section we will present a brief overview of the different methods of proof of the Malgrange-Ehrenpreis theorem.

2.1 Classification of existing proofs

According to [10], the existing proofs of the Malgrange-Ehrenpreis theorem can be classified into three categories:

(a) non-constructive proofs using Hahn-Banach theorem;
(b) constructive proofs by means of explicit formulae;
(c) proofs by solution of a division problem.

2.1.1 Original proofs of B. Malgrange and L. Ehrenpreis

The existence of fundamental solutions for every linear partial differential equation with constant coefficients was first proved independently by Bernard Malgrange [9, Th. 1, p. 28] and Leon Ehrenpreis [3, Th. 6, p. 892] in 1954–1956. We describe the general procedure of the original proofs. Consider the linear functional

$$F : P^*(\partial)\mathcal{D} \longrightarrow \mathbb{C} : P^*(\partial)\phi \longmapsto \phi(0) \tag{2.1}$$

on the subspace $P^*(\partial)\mathcal{D} = \{P^*(\partial)\phi : \phi \in \mathcal{D}\}$ of $\mathcal{D}$. The key step of the proof consists in showing the continuity of the linear functional F, for once it is shown that F is continuous with respect to the topology induced by $\mathcal{D}$, the Hahn-Banach theorem implies the existence of a continuous linear extension E on the space of all tested functions, i.e. $E \in \mathcal{D}'$. Then E is a fundamental solution of $P(\partial)$, as:

$$\langle \phi, P(\partial)E\rangle = \langle P^*(\partial)\phi, E\rangle = \langle P^*(\partial)\phi, F\rangle = \phi(0) = \langle \delta, \phi\rangle \tag{2.2}$$

Thus, it remains to prove the continuity of F and this task is accomplished by the use of an "a-priori inequality" (cf. [7, pp. 17-18]), such as:

Theorem 2.1 (Hörmander's inequality)**.** *Let $P(\partial)$ be a non-zero linear differential operator with constant coefficients. For every bounded domain $\Omega \in \mathbb{R}^n$, there exists a constant $C > 0$, such that for every $\phi \in \mathcal{D}(\Omega)$:*

$$\|\phi\|_2 \leq C\|P(\partial)\phi\|_2 \tag{2.3}$$

The continuity of the functional F in the L^2 norm follows by applying the Hörmander's inequality to the adjoint operator $P^*(\partial)$ [13, p. 520].

The original methods of proof of Theorem 1.1 used an argumentation based on estimations. The Hahn-Banach theorem guarantees the existence of a linear extension E of F, however E is not uniquely determined and thus an explicit representation of E cannot be obtained. Explicit formulae yielding fundamental solutions were known for certain classes of differential operators (e.g., for hyperbolic, elliptic, and, more generally, hypoelliptic[1] operators, cf. [6, p. 223]), so the question of explicit formulae for general linear operators with constant coefficients quickly arose. This led to proofs relying on formulae for a fundamental solution.

2.1.2 Constructive proofs

Lars Hörmander was the first to give an "explicit" general formula [6, p.67] for fundamental solutions of linear differential operators with constant coefficients. In his Ph.D. thesis [6, p. 223] he generalized the approach used for hypoelliptic operators.

The method of construction of fundamental solutions via the so-called "Hörmander's staircase" involves partitions of unity.

For $P(x) \in \mathbb{C}[x]$ with $\deg P = m$, and $\eta \in \mathbb{R}^n$ with $P_m(\eta) \neq 0$, we choose measurable functions $\chi_k : \mathbb{R}^n \to 0, 1, k = 0, \ldots, m$ such that:

(i) $\chi_k(\xi + \lambda\eta) = \chi_k(\xi)$ for all $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, and $k = 0, \ldots, m$;

(ii) $\displaystyle\sum_{k=0}^{m} \chi_k(\xi) = 1$ for all $\xi \in \mathbb{R}^n$;

(iii) $\exists\, C > 0$ such that $\chi_k(\xi) \neq 0 \Longrightarrow |P(i\xi + k\eta)| > C$ for all $\xi \in \mathbb{R}^n$ and $k = 0, \ldots, m$.

The set of $(m + 1)$ functions ξ_k, $k = 0, \ldots, m$, with the properties (i), (ii), (iii) form the

[1] A differential operator L with C^∞ coefficients is called *hypoelliptic* if any distribution u on an open set Ω such that Lu is C^∞ must be itself C^∞ on Ω.

Hörmander's staircase. Then $E \in \mathcal{D}'(\mathbb{R}^n)$ given by:

$$E := \sum_{k=0}^{m} e^{kx\eta} \mathcal{F}^{-1}_{\xi \to x} \left(\frac{\chi_k(\xi)}{P(i\xi + k\eta)} \right) \tag{2.4}$$

is a fundamental solution of $P(\partial)$ [10]. The existence of the Hörmander's staircase is proved by applying Dirichlet's pigeonhole principle to the set of zeros of the polynomial $z \longmapsto P(i\xi + z\eta)$ in one complex variable z (cf. [11, Tm. 2.3]).

We verify that E is indeed a fundamental solution of $P(\partial)$:

$$\begin{aligned} P(\partial)E &= \sum_{k=0}^{m} e^{kx\eta} P(\partial + k\eta) \mathcal{F}^{-1}_{\xi \to x} \left(\frac{\chi_k(\xi)}{P(i\xi + k\eta)} \right) && (2.5a) \\ &= \sum_{k=0}^{m} e^{kx\eta} \mathcal{F}^{-1}_{\xi \to x}(\chi_k(\xi)) && (2.5b) \\ &= \sum_{k=0}^{m} \mathcal{F}^{-1}_{\xi \to x}(\chi_k) && (2.5c) \\ &= \delta, && (2.5d) \end{aligned}$$

where in (2.5c) we used property (i) and the fact that $f(x\eta)\mathcal{F}^{-1}_{\xi \to x}(\chi_k(\xi)) = f(0)\mathcal{F}^{-1}_{\xi \to x}(\chi_k(\xi))$ for $f \in C^\infty(\mathbb{R})$ [11].

However, the Hörmander construction is of limited practical use, since the solution defined by (2.4) is often difficult to compute and it does not coincide with physically relevant fundamental solution (cf. Example 2.1). Thereafter we consider more sophisticated formulae for fundamental solutions.

Example 2.1. The unique fundamental solution of Laplace's equation, which vanishes at infinity, is:

$$E = \frac{1}{n(n-2)\alpha_n |x|^{n-2}} = -\frac{\Gamma(\frac{n}{2} - 1)}{4\pi^{n/2}|x|^{n-2}} \tag{2.6}$$

On the other hand, the solution obtained by the Hörmander's formula is:

$$\begin{aligned} F &= e^{-2\varepsilon x_n/3} \mathcal{F}^{-1}_{\xi \to x} \left(\frac{\chi_0(\xi)}{(i\xi_n - \frac{2\varepsilon}{3})^2 - |\xi'|^2} \right) + \mathcal{F}^{-1}_{\xi \to x} \left(\frac{\chi_1(\xi)}{-|\xi|^2} \right) && (2.7) \\ &= F + \frac{(2\pi)^{(1-n)/2}}{2|x'|^{(n-3)/2}} \int_0^{\varepsilon/3} \rho^{(n-3)/2} e^{-\rho x_n} J_{(n-3)/2}(\rho|x'|) \, \mathrm{d}\rho, && (2.8) \end{aligned}$$

where $\varepsilon > 0, \chi_1(\xi) = Y(\varepsilon - |\xi'|) \cdot Y(|\xi'| - \frac{\varepsilon}{3}), \chi_0 = 1 - \chi_1, \chi_2 = 0$. The equation (2.8) follows

from the Bochner's formula (cf. [14, (VII, 7; 22), p. 259]). The details of these calculations can be found in [11] and [16].

The staircase construction requires some knowledge of the location of zeros of the polynomials $z \longmapsto P(i\xi + z\eta)$ for $\xi, \eta \in \mathbb{R}^n$. Heinz König proposed to replace the bounded measurable functions $\xi \longmapsto \frac{\chi_k(\xi)}{P(i\xi+k\eta)}$ ($k = 0, \ldots, m, \eta \in \mathbb{R}^n$ fixed) by the functions $\xi \longmapsto \frac{\overline{P(i\xi+\varepsilon\eta)}}{P(i\xi+\varepsilon\eta)}$ of unit modulus with $\eta \in \mathbb{T}^n, \varepsilon > 0$ fixed, cf. [8]. In comparison with Hörmander's staircase, König's method involves n additional integration over the n-dimensional torus $\mathbb{T}^n$.

Theorem 2.2. *Let $P(\partial) \in \mathbb{C}[\partial_1, \ldots, \partial_n]$. If $P \not\equiv 0$, $m := \deg P$, and $\varepsilon > 0$, then the distribution E defined by:*

$$E := \frac{1}{\varepsilon^m \|P_m(\eta)\|^2} \int_{\mathbb{T}^n} e^{\varepsilon\eta x} P_m(\eta) \mathcal{F}^{-1}_{\xi\to x}\left(\frac{\overline{P(i\xi+\varepsilon\eta)}}{P(i\xi+\varepsilon\eta)}\right) \times \frac{\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_n}{(2\pi i\eta_1)\cdots(2\pi i\eta_n)} \tag{2.9}$$

is a fundamental solution of $P(\partial)$ which fulfills $E/\cosh(\sqrt{n}\varepsilon|x|) \in \mathcal{S}'(\mathbb{R}^n_x)$. For $m \in \mathbb{N}$ fixed, E continuously depends on the coefficients of $P(\partial) \in \mathbb{C}[\partial]_m \setminus \mathbb{C}[\partial]_{m-1}$.

The proof of this theorem is omitted. Instead, we present a simpler proof by Ortner and Wagner [10, pp. 4–5], based on König's approach. A simpler representation of a fundamental solution is valid provided that $\eta \in \mathbb{R}^n$ with $P_m(\eta) \neq 0$ is fixed.

Theorem 2.3. *Let $P(\partial) \in \mathbb{C}[\partial]$, $m := \deg P$, and $\eta \in \mathbb{R}^n$ with $P_m(\eta) \neq 0$. Then the distribution E given by:*

$$E := \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \mathcal{F}^{-1}_{\xi\to x}\left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)}\right) \frac{\mathrm{d}\lambda}{2\pi i\lambda} \tag{2.10}$$

is a fundamental solution of $P(\partial)$ which fulfills $E/\cosh(\eta x) \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. Following the argument in [10], we observe that for a fixed $\lambda \in \mathbb{C}$, $\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)} \in L^\infty(\mathbb{R}^n_\xi)$. Furthermore, Lebesgue's dominated convergence theorem shows that the mapping:

$$\mathbb{T}^1 \longrightarrow \mathcal{S}'(\mathbb{R}^n_\xi) : \lambda \longmapsto \frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)} \tag{2.11}$$

is continuous. Since a continuous function with values in $\mathcal{D}'(\mathbb{R}^n)$ can be integrated over a compact set, E is well-defined, and the distribution $E/\cosh(\eta x)$ is temperate.

Moreover,

$$\begin{aligned}
P(\partial)E &= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m P(\partial) \left[e^{\lambda\eta x} \mathcal{F}^{-1}_{\xi\to x} \left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)} \right) \right] \frac{\mathrm{d}\lambda}{2\pi i\lambda} \\
&= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \left[P(\partial+\lambda\eta) \mathcal{F}^{-1}_{\xi\to x} \left(\frac{\overline{P(i\xi+\lambda\eta)}}{P(i\xi+\lambda\eta)} \right) \right] \frac{\mathrm{d}\lambda}{2\pi i\lambda} \\
&= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \mathcal{F}^{-1}_{\xi\to x} (\overline{P(i\xi+\lambda\eta)}) \frac{\mathrm{d}\lambda}{2\pi i\lambda} \\
&= \frac{1}{\overline{P_m(\eta)}} \int_{\mathbb{T}^1} \lambda^m e^{\lambda\eta x} \overline{P(\partial+\lambda\eta)}\delta \frac{\mathrm{d}\lambda}{2\pi i\lambda} \qquad (2.12)
\end{aligned}$$

By Taylor's theorem,

$$\overline{P(\partial+\lambda\eta)}\delta = \overline{\lambda^m}\,\overline{P_m(\eta)}\delta + \sum_{k=0}^{m-1} \overline{\lambda^k} Q_k(\partial)\delta, \qquad (2.13)$$

for some polynomials Q_k. By the residue theorem, the integrals over the terms with the factors $\overline{\lambda^k}$, $k = 0, \ldots, m-1$ vanish, and the integral over the leading terms yields δ. Thus E is a fundamental solution of $P(\partial)$. This completes the proof. □

2.1.3 Solution of a division problem

The third class of proofs shows the existence of a temperate fundamental solution, i.e. $E \in \mathcal{S}'(\mathbb{R}^n)$ with $P(\partial)E = \delta$. This is equivalent to solving the "division problem" $P(i\xi)\mathcal{F}E = 1$ or $E = \mathcal{F}^{-1}(P(i\xi)^{-1})$. This can be done by either estimating the function $|P(i\xi)|^{-1}$ near the variety $\{\xi \in \mathbb{R}^n : P(i\xi) = 0\}$ or by extending the distribution-valued holomorphic function

$$\{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} \longrightarrow \mathcal{S}' : \lambda \longmapsto |P(i\xi)|^{\lambda} \qquad (2.14)$$

A special method of extension was developed by Bernstein around 1972 in [1] and [2]. The construction of Bernstein's fundamental solution depends heavily on the study of modules over the n-th complex Weyl algebra[2]. He showed that there exist polynomials $b(\lambda) \in \mathbb{C}[\lambda]$ and $Q(\lambda, x, \partial) \in \mathbb{C}[\lambda, x, \partial]$ such that the equation $Q(\lambda, x, \partial)P^{\lambda+1} = b(\lambda)P^{\lambda}$ holds in the algebraic sense, i.e., with the definition $\partial_i P^{\mu} := \mu\partial P^{\mu-1}$ for $\mu \in \mathbb{C}$ and $i = 1, \ldots, n$. However, Bernstein's proof of the Malgrange-Ehrenpreis theorem, as well as the other proofs of this kind, is not explicit, since the existence of the polynomials b and Q is shown by indirect arguments.

[2] The n-th Weyl algebra is the ring of differential operators with polynomial coefficients in n variables, generated by x_i and ∂_i.

3 Conclusion

We have studied various methods of proofs of the Malgrange-Ehrenpreis theorem. The most useful proofs not only show the existence of fundamental solutions, but also provide explicit formulae to find them.

With a fundamental solution E in hand, we can solve the equation $P(\partial)u = f$ not only when $f \in \mathcal{D}$ but f is any distribution with compact support. Indeed, if $f \in \mathcal{D}'$, we have:

$$P(\partial)(E * f) = P(\partial)E * f = \delta * f = f \tag{3.1}$$

Moreover, fundamental solutions allow us to study the regularity properties of solutions of differential equations.

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