Notes on Basic Hodge Theory

Yang Guo
260313651

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1 Introduction

Hodge theory, named after W. V. D. Hodge, is originally formulated for the de Rham complex. Practically ,it can be used to study Riemannian and Kahler manifold and algebraic geometry of complex projective varieties . It is related to the study of nonlinear PDEs. Also, the famous Hodge Conjecture is among one of the Clay Mathematics Institute's Millennium Prize Problems.
In this note,we will introduce the basics of Hodge theory. M will always denote compact oriented Reimannian manifold of dimension n.we will define the notion of Lapalace-Beltrami operator,which is the generalization of the classical Δ on differential forms.We will prove the Hodge decomposition theorem,which says that this equationδhas a solution w in the smooth p-forms on M if and only if the p-form α is orthogonal(w.r.t some inner product on epm to the space of harmonic p-forms.From it, we will show that there exists a unique harmonic form in each de Rham cohomology class.

2 Laplace-Beltrami Operator

First , we will define the operator $*$. Let V be an n dimensional inner product space , we will extend the inner product on V to all of $\Lambda(V)$, which is the space of all differential forms on V. The inner product is defined such that , the inner product of elements which are homogeneous of different degrees equals to zero, and for same degree

$$< w_1 \wedge \cdots \wedge w_p, v_1 \wedge \cdots \wedge v_p >= det < w_i, v_j > .$$

and extending binearly to all of $\Lambda_p(V)$, the space of differential forms of homogeneous degree p , hence extending to $\Lambda(V)$. For $\{e_1, \cdots, e_n\}$ an orthonormal basis of V , we have $\{e_{i_1} \wedge \cdots \wedge e_{i_r}; i_1 \leq \cdots i_r\}$ is an orthonormal basis for $\Lambda(V)$. Note that since $\Lambda_n(V)$ is one-dimensional, $\Lambda_n(V) - \{0\}$ has two component. We define *orientation* on V to be a choice of component of $\Lambda_n(V) - \{0\}$. For V an oriented inner product space , there is a linear transformation :

$$* : \Lambda(V) \to \Lambda(V),$$

called *star* which is defined as follows :

$$*(1) = \pm e_1 \wedge \cdots \wedge e_n, *(e_1 \wedge \cdots \wedge e_n) = \pm 1.$$

$$*(e_1 \wedge \cdots \wedge e_p) = \pm e_{p+1} \wedge \cdots \wedge e_n,$$

where one takes $+$ if $e_1 \wedge \cdots \ wedgee_n$ lies in the component of $\Lambda_n(V) - \{0\}$ determined by the orientation and $-$ otherwise.
Now we observe that

$$*\Lambda_p(V) \to \Lambda_{n-p}(V).$$

And on $\Lambda_p(V)$,

$$** = (-1)^{p(n-1)}.$$

Also, one can easily verify for any $v, w \in \Lambda_p(V)$,

$$< v, w >= *(w \wedge *v) = *(v \wedge *w).$$

We also define an operator δ from p- form to $p-1$- form by

$$\delta = (-1)^{n(p+1)+1} * d * .$$

On 0-form, we define it to be the zero linear functional. The *Laplace-Beltrami* Δ is defined as

$$\Delta = \delta d + d\delta,$$

which is linear on space of smooth p-froms, $E^p(M)$. In particular for $p = 0$, on $E^0(M) = C^\infty(M)$, $\Delta = \sum_{i=1}^{i=n}(-1)\frac{\partial^2}{\partial x_i^2}$. Also, one can check that

$$*\Delta = \Delta * .$$

We define the inner product on $E^p(M)$ of smooth p-forms on M by:

$$< \alpha, \beta >= \int_M \alpha \wedge *\beta$$

where $\alpha, \beta \in E^p(M)$. and denote norm w.r.t this inner product as $\| \|$. It is an exercise to show that this is in fact a well defined inner product. We can extend the inner product on $\sum_{p=0}^{p=n} E^p(M)$ by requiring that different $E^p(M)$ to be orthogonal analogous to what we do on $\Lambda_p(V)$.

Lemma 1 *δ is the adjoint of d on $\sum_{p=0}^{p=n} E^p(M)$, that is $< d\alpha, \beta >=< \alpha, \delta\beta >$*

proof:
It is enough to prove the case where α is a $p-1$ form and β is a p form and extend by linearity and orthogonality to all $\sum_{p=0}^{p=n} E^p(M)$.

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d * \beta = d\alpha \wedge *\beta - \alpha \wedge *\delta\beta.$$

Now we integrate over M and use Stoke's theorem ,we can obtain

$$0 = \int_M (d\alpha \wedge *\beta - \alpha \wedge *\delta\beta) =< d\alpha, \beta > - < \alpha, \delta\beta >$$

Hence our result.

Corollary 2 Δ *is self adjoint , that is* $< \Delta\alpha, \beta >=< \alpha, \Delta\beta >$ *where* $\alpha, \beta \in E^p(M)$

Proposition 3 $\Delta\alpha = 0$ *if and only if* $d\alpha = 0$ *and* $\delta\alpha = 0$

proof
Clearly $\Delta\alpha = 0$ if $d\alpha = 0$ and $\delta\alpha = 0$ by definition.
On the other hand,

$$< \Delta\alpha, \alpha >=< (d\delta + \delta d)\alpha, \alpha >=< \delta\alpha, \delta\alpha > + < d\alpha, d\alpha > .$$

So if $\Delta\alpha = 0$, $d\alpha$ and $\delta\alpha$ must be zero

Corollary 4 *The only harmonic functions on a compact , connected ,oriented ,Riemannian manifold are the constant functions*

3 The Hodge Theorem and its Consequences

We use Δ^* to denote the adjoint of Δ which is just Δ itself . We use this notation for symbolical convenience. We are interested in finding necessary and sufficient condition for existence of solution to $\Delta\omega = \alpha$ as remarked in the introduction. To proceed , we first try to find its weak solution. Note that

$$< \Delta\omega, \varphi >=< \alpha, \varphi > \text{ for all} \varphi \in E^p(M)$$

from it,

$$< \omega, \Delta^*\varphi >=< \alpha, \varphi > \text{ for all} \varphi \in E^p(M)$$

We can see that ω determines a bounded linear fucntional l on $E^p(M)$:

$$l(\beta) =< \omega, \beta > .$$

Here,

$$l(\Delta^*) =< \alpha, \varphi > \text{ for all} \varphi \in E^p(M)$$

Definition 5 *We denote* $H^p = \{\omega \in E^p(M) : \Delta\omega = 0\}$. *The elements in* H^p *are called harmonic p-forms.*

We notice that the ordniary solution of $\Delta\omega = \alpha$ determines a weak solution l, we also have the converse that each weak solution l of $\Delta\omega = \alpha$ is represented by a smooth form ω such that $l(\beta) =< \omega, \beta >$ holds. Now ,

$$< \Delta\omega, \beta >=< \omega, \Delta^* >= l(\Delta^*\beta) =< \alpha, \beta >$$

for all $\beta \in E^p(M)$, which shows $\Delta\omega = \alpha$ holds. To summarize

Theorem 6 *Let $\alpha \in E^p(M)$ and let l be the weak solution of $\Delta\omega = \alpha$. Then there exist $\omega \in E^p(M)$ such that $l(\beta) =< \omega, \beta >$ for every $\beta \in E^p(M)$. Also , $\Delta\omega = \alpha$.*

Theorem 7 *For any sequence $\{\alpha_n\}$ of smooth p-forms on M such that $\|a_n\| \leq c$ and $|\Delta a_n| \leq c$ for all n and for some $c > 0$. Then a subsequence of $\{a_n\}$ is a Cauchy sequence in $E^p(M)$.*

We will not prove these two theorems. The proofs are very technical. The outline is to employ the Fourier series. And one need some result in Soblev space and functional analysis , (e.g Soblev lemma,Rellich lemma and Peter-Paul-Inequality) . Also ,we need to reduce the problem to the periodic case and show the operator is elliptic.

Theorem 8 *For each interger p with $0 \leq p \leq n$, H^p is finite dimensional , with the following orthogonal direct su decomposition:*

$$E^p(M) = \Delta E^p \oplus H^p = d\delta(E^p) \oplus \delta d(E^p) \oplus H^p = d(E^{p-1}) \oplus \delta(E^{p+1}) \oplus H^p.$$

Consequently , we have from the decomposition , the equation $\Delta\omega = \alpha$ has a solution $\omega \in E^p(M)$ if and only if the p-form α is orthogonal to the space of harmonic p-forms, h^p.

proof
First, we can show that H^p is finite dimensional. Since if not, there exist an infinite orthonormal sequence , by previous theorem, this orthonormal sequence contain a Cauchy subsequence , which contradicts the orthonormality.
It is also enough to prove the first = for the identity, the rest follows from previous computations. For any $\alpha \in E^p(M)$, we can write uniquely:

$$\alpha = \beta + \sum_{i=1}^{l} < \alpha, \omega_i > \omega_i.$$

where $\beta \in (H^p)^\perp$. Now $E^p(M) = (H^p)^\perp \oplus H^p$, we will show that $(H^p)^\perp = \Delta(E^p)$. We have $\Delta(E^p) \subset (H^p)^\perp$. For $\omega \in E^p$ and $\alpha \in H^p$,

$$< \Delta\omega, \alpha >=< \omega, \Delta >= 0.$$

On the other hand, we want to show $(H^p)^\perp \subset \Delta(E^p)$. To do this, we first prove the fact that there is a constant c such that

$$|\beta| \leq |\Delta\beta| , \forall\beta \in (H^p)^\perp.$$

If this were not true, we would have a sequence $\beta_j \in (H^p)^\perp$ with $|\beta_j| = 1$ and $|\Delta\beta_j| \to 0.$, then by previous theorem, there is a subsequence of β_j, wlog suppose it's $\{\beta_j\}$ itself, is Cauhcy . $lim_{j\to\infty} < \beta_j,$
$psi >$ exist for every $\psi \in E^p$. We define the linear functional on E^p

$$l(\psi) = \lim_{j\to\infty} < \beta_j, \psi >$$

Note that l is bounded and $l(\Delta\varphi) = \lim_{j\to\infty} < \beta_j, \Delta\varphi >=< \Delta\beta_j, \varphi >= 0$. So l is the weak solution of $\Delta\beta = 0$. By previous theorem, there exists $\beta \in E^p(M)$ such that $l(\psi) =< \beta, \psi >$. And $\beta_j \to \beta$. Because $|\beta_j| = 1$ and $\beta_j \in (H^p)^\perp \Rightarrow |\beta| = 1, \beta \in (H^p)^\perp$. But by previous theorem, $\Delta\beta = 0$ and $\beta \in H^p$ which is a contradiction.
Now , we can use this to prove $(H^p)^\perp \subset \Delta E^p$. Let $\alpha \in (H^p)^\perp$, we redefine our l to be

$$l(\Delta\varphi) =< \alpha, \varphi >, \forall\varphi \in E^p.$$

It is easy to check l is a well defined bounded linear functional on $\Delta(E^p)$. Let $\varphi \in E^p(M)$ and $\psi = \varphi - H(\varphi)$, here $H(\varphi)$ is the harmonic part of φ. We obtain

$$|l(\Delta\varphi)| = |l(\Delta\psi)| = | < \alpha, \psi >\leq |\alpha|\,|\,|\psi| \leq c\,|\alpha|\,|\Delta\psi| = c\,|\alpha|\,|\Delta\varphi|$$

By Hahn Banach theorem, l extends to a bounded linear functional on $E^p(M)$. l is a weak solution of $\Delta\omega = \alpha$. Hence we get the inclusion. $(H^p)^\perp = \Delta(E^\perp)$ completes our proof of Hodge's decomposition theorem.

Remark 9 *(Hahn Banach) Let p be a finite convex functional defined on a linear space L , and let L_0 be the subspace of L. Suppose that f_0 is a linear functional on L_0 satisfying the condition $f_0(x) \leq p(x)$ on L_0. Then f_0 can be extended to a linear functional L satisfying $f(x) \leq p(x)$ on the whole space L.*

Definition 10 *We can also define Green's operator , $G : E^p(M) \to (H^p)^\perp$ by setting $G(\alpha)$ equal to the unique solution of $\Delta\omega = \alpha - H(\alpha) \in (H^p)^\perp$. One can check that G is a bounded self adjoint operator which takes bounded sequences into sequences with Cauchy subsequences. Also, G commutes with d, δ, and Δ. We have in fact a stronger result:*

Proposition 11 *For any linear operator which T such that $T\Delta = \Delta T$.*

proof
We let $T : E^p(M) \to E^q(M)$ be our linear operator such that $T\Delta = \Delta T$. Denote $\pi_{(H^p)^\perp}$ to be the projection map on to $(H^p)^\perp$. By definition ,$G = (\Delta|(H^p)^\perp)^{-1} \circ \pi_{(H^p)^\perp}$. From hodge decomposition theorem along with identity $T\Delta = \Delta T$, we have $T(H^p) \subset H^q$ and $T((H^p)^\perp) \subset (H^q)^\perp$. We can now get

$$T \circ \pi_{(H^p)}{}^\perp = \pi_{(H^p)}{}^\perp \circ T \Rightarrow$$

$$T \circ (\Delta|(H^p)^\perp) = (\Delta|(H^p)^\perp) \circ T$$

and finally,

$$T \circ (\Delta|(H^p)^\perp)^{-1} = (\Delta|(H^p)^\perp)^{-1} \circ T.$$

This would imply directly that G commutes with T.

Definition 12 *A sequence of vector spaces and maps* (V_i, d_i)*, where* $d_i : V_i \to V_{i+1}$*, is called a complex if* $d_{i+1} \circ d_i = 0$*. Consider the complex*

$$\xrightarrow{d_{p-2}} \Omega^{p-1}(M) \xrightarrow{d_{p-1}} \Omega^p(M) \xrightarrow{d_p} \Omega^{p+1}(M) \xrightarrow{d_{p+1}}$$

and define $H^p_{dR}(M) = \frac{Ker(d_i)}{Im(d_{i-1})}$

Theorem 13 *Each de Rham cohomology class on a compact oriented Riemannian manifold* M *contains a unique harmonic representative.*

proof
For any p-form on M, we have

$$\alpha = d\delta G\alpha + \delta dG\alpha + H\alpha = d\delta G\alpha + \delta Gd\alpha + H\alpha$$

by commutativity of Green's operator. if α is a closed form ,

$$\alpha = d\delta G\alpha + H\alpha.$$

Here $H\alpha$ is a harmonic p-form in the same deRham cohomology class as α. If two harmonic forms α_1 and α_2 differ by an exact form $d\beta$, then

$$0 = d\beta + (\alpha_1 - \alpha_2).$$

Note that $< d\beta, \alpha_1 - \alpha_2 >=< \beta, \delta\alpha_1 - \delta\alpha_2 >=< \beta, 0 >= 0$. We have that $d\beta = 0$ and $\alpha_1 = \alpha_2$. Hence ,we showed that there is a unique harmonic form in each de Rham cohomology class.

Remark 14 *Recall that a closed form* α *is a differential form whose exterior derivative* $d\alpha$ *is zero. A exact form* α *is a differential form such that there exist a form* β *,* $d\beta = \alpha$

4 Reference

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