Notes on Basic Hodge Theory

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1 Introduction

Hodge theory, named after W. V. D. Hodge, is originally formulated for the de Rham complex. Practically, it can be used to study Riemannian and Kahler manifold and algebraic geometry of complex projective varieties. It is related to the study of nonlinear PDEs. Also, the famous Hodge Conjecture is among one of the Clay Mathematics Institute’s Millennium Prize Problems.

In this note, we will introduce the basics of Hodge theory. \( M \) will always denote compact oriented Riemannian manifold of dimension \( n \), we will define the notion of Laplace-Beltrami operator, which is the generalization of the classical \( \Delta \) on differential forms. We will prove the Hodge decomposition theorem, which says that this equation has a solution \( w \) in the smooth \( p \)-forms on \( M \) if and only if the \( p \)-form \( \alpha \) is orthogonal (w.r.t some inner product on \( \text{epm} \)) to the space of harmonic \( p \)-forms. From it, we will show that there exists a unique harmonic form in each de Rham cohomology class.

2 Laplace-Beltrami Operator

First, we will define the operator \( * \). Let \( V \) be an \( n \)-dimensional inner product space, we will extend the inner product on \( V \) to all of \( \Lambda(V) \), which is the space of all differential forms on \( V \). The inner product is defined such that, the inner product of elements which are homogeneous of different degrees equals to zero, and for same degree

\[
<w_1 \wedge \cdots \wedge w_p, v_1 \wedge \cdots \wedge v_p> = \det <w_i, v_j>.
\]

and extending bilinearly to all of \( \Lambda_p(V) \), the space of differential forms of homogeneous degree \( p \), hence extending to \( \Lambda(V) \). For \( \{e_1, \cdots, e_n\} \) an orthonormal basis of \( V \), we have \( \{e_{i_1} \wedge \cdots \wedge e_{i_r}; i_1 \leq \cdots \leq i_r\} \) is an orthonormal basis for \( \Lambda(V) \).

Note that since \( \Lambda_n(V) \) is one-dimensional, \( \Lambda_n(V) - \{0\} \) has two component. We define orientation on \( V \) to be a choice of component of \( \Lambda_n(V) - \{0\} \). For \( V \) an oriented inner product space, there is a linear transformation:

\[
* : \Lambda(V) \to \Lambda(V),
\]

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called *star* which is defined as follows:

\[(1) = \pm e_1 \wedge \cdots \wedge e_n, \quad *(e_1 \wedge \cdots \wedge e_n) = \pm 1.\]

\[*(e_1 \wedge \cdots \wedge e_p) = \pm e_{p+1} \wedge \cdots \wedge e_n,\]

where one takes + if \(e_1 \wedge \cdots \wedge e_n\) lies in the component of \(\Lambda_n(V) - \{0\}\) determined by the orientation and otherwise.

Now we observe that

\[\star \Lambda_p(V) \to \Lambda_{n-p}(V).\]

And on \(\Lambda_p(V),\)

\[** = (-1)^{p(n-1)}.\]

Also, one can easily verify for any \(v, w \in \Lambda_p(V),\)

\[<v, w> = *(w \wedge *v) = *(v \wedge *w).\]

We also define an operator \(\delta\) from \(p\)-form to \(p-1\)-form by

\[\delta = (-1)^{n(p+1)+1} \ast d \ast.\]

On 0-form, we define it to be the zero linear functional. The Laplace-Beltrami \(\Delta\) is defined as

\[\Delta = \delta d + d \delta,\]

which is linear on space of smooth \(p\)-forms, \(E^p(M).\) In particular for \(p = 0,\) on \(E^0(M) = C^\infty(M),\)

\[\Delta = \sum_{i=1}^{n} (-1) \frac{\partial^2}{\partial x_i^2}.\]

Also, one can check that

\[\star \Delta = \Delta \ast.\]

We define the inner product on \(E^p(M)\) of smooth \(p\)-forms on \(M\) by:

\[<\alpha, \beta> = \int_M \alpha \wedge \ast \beta\]

where \(\alpha, \beta \in E^p(M)\) and denote norm w.r.t this inner product as \(\|\|\). It is an exercise to show that this is in fact a well defined inner product. We can extend the inner product on \(\sum_{p=0}^{n} E^p(M)\) by requiring that different \(E^p(M)\) to be orthogonal analogous to what we do on \(\Lambda_p(V)\).

**Lemma 1** \(\delta\) is the adjoint of \(d\) on \(\sum_{p=0}^{n} E^p(M),\) that is \(<d \alpha, \beta> = <\alpha, \delta \beta>\)
proof:  
It is enough to prove the case where \( \alpha \) is a \( p-1 \) form and \( \beta \) is a \( p \) form and extend by linearity and orthogonality to all \( \sum_{p=0}^{n} E^p(M) \).

\[
d(\alpha \wedge * \beta) = d\alpha \wedge * \beta + (-1)^{p-1} \alpha \wedge d* \beta = d\alpha \wedge * \beta - \alpha \wedge \delta \beta.
\]

Now we integrate over \( M \) and use Stoke’s theorem, we can obtain

\[
0 = \int_M (d\alpha \wedge * \beta - \alpha \wedge \delta \beta) = \langle d\alpha, \beta \rangle - \langle \alpha, \delta \beta \rangle
\]

Hence our result.

**Corollary 2** \( \Delta \) is self adjoint, that is \( \langle \Delta \alpha, \beta \rangle = \langle \alpha, \Delta \beta \rangle \) where \( \alpha, \beta \in E^p(M) \)

**Proposition 3** \( \Delta \alpha = 0 \) if and only if \( d\alpha = 0 \) and \( \delta \alpha = 0 \)

**proof**

Clearly \( \Delta \alpha = 0 \) if \( d\alpha = 0 \) and \( \delta \alpha = 0 \) by definition.

On the other hand,

\[
\langle \Delta \alpha, \alpha \rangle = \langle (d \delta + \delta d) \alpha, \alpha \rangle = \langle \delta \alpha, \delta \alpha \rangle + \langle d\alpha, d\alpha \rangle.
\]

So if \( \Delta \alpha = 0 \) , \( d\alpha \) and \( \delta \alpha \) must be zero

**Corollary 4** The only harmonic functions on a compact, connected, oriented, Riemannian manifold are the constant functions

### 3 The Hodge Theorem and its Consequences

We use \( \Delta^* \) to denote the adjoint of \( \Delta \) which is just \( \Delta \) itself. We use this notation for symbolical convenience. We are interested in finding necessary and sufficient condition for existence of solution to \( \Delta \omega = \alpha \) as remarked in the introduction.

To proceed, we first try to find its weak solution. Note that

\[
\langle \Delta \omega, \varphi \rangle = \langle \alpha, \varphi \rangle \text{ for all } \varphi \in E^p(M)
\]

from it,

\[
\langle \omega, \Delta^* \varphi \rangle = \langle \alpha, \varphi \rangle \text{ for all } \varphi \in E^p(M)
\]

We can see that \( \omega \) determines a bounded linear functional \( l \) on \( E^p(M) \) :

\[
l(\beta) = \langle \omega, \beta \rangle.
\]

Here,

\[
l(\Delta^*) = \langle \alpha, \varphi \rangle \text{ for all } \varphi \in E^p(M)
\]

**Definition 5** We denote \( H^p = \{ \omega \in E^p(M) : \Delta \omega = 0 \} \). The elements in \( H^p \) are called harmonic \( p \)-forms.
We notice that the ordinary solution of $\Delta \omega = \alpha$ determines a weak solution $l$, we also have the converse that each weak solution $l$ of $\Delta \omega = \alpha$ is represented by a smooth form $\omega$ such that $l(\beta) = \langle \omega, \beta \rangle$ holds. Now,

$$
\langle \Delta \omega, \beta \rangle = \langle \omega, \Delta^* \beta \rangle = l(\Delta^* \beta) = \langle \alpha, \beta \rangle
$$

for all $\beta \in E^p(M)$, which shows $\Delta \omega = \alpha$ holds. To summarize

**Theorem 6** Let $\alpha \in E^p(M)$ and let $l$ be the weak solution of $\Delta \omega = \alpha$. Then there exist $\omega \in E^p(M)$ such that $l(\beta) = \langle \omega, \beta \rangle$ for every $\beta \in E^p(M)$. Also, $\Delta \omega = \alpha$.

**Theorem 7** For any sequence $\{\alpha_n\}$ of smooth $p$-forms on $M$ such that $\|\alpha_n\| \leq c$ and $|\Delta \alpha_n| \leq c$ for all $n$ and for some $c > 0$. Then a subsequence of $\{\alpha_n\}$ is a Cauchy sequence in $E^p(M)$.

We will not prove these two theorems. The proofs are very technical. The outline is to employ the Fourier series. And one need some result in Soblev space and functional analysis , (e.g Soblev lemma,Rellich lemma and Peter-Paul-Inequality ) . Also ,we need to reduce the problem to the periodic case and show the operator is elliptic.

**Theorem 8** For each interger $p$ with $0 \leq p \leq n$, $H^p$ is finite dimensional , with the following orthogonal direct su decomposition:

$$
E^p(M) = \Delta E^p \oplus H^p = d\delta(E^p) \oplus \delta d(E^p) \oplus H^p = d(E^{p-1}) \oplus \delta(E^{p+1}) \oplus H^p.
$$

Consequently , we have from the decomposition , the equation $\Delta \omega = \alpha$ has a solution $\omega \in E^p(M)$ if and only if the $p$-form $\alpha$ is orthogonal to the space of harmonic $p$-forms, $h^p$.

**proof**
First, we can show that $H^p$ is finite dimensional. Since if not, there exist an infinite orthonormal sequence , by previous theorem, this orthonormal sequence contain a Cauchy subsequence , which contradicts the orthonormality.

It is also enough to prove the first = for the identity, the rest follows from previous computations. For any $\alpha \in E^p(M)$, we can write uniquely:

$$
\alpha = \beta + \sum_{i=1}^l \langle \alpha, \omega_i \rangle \omega_i.
$$

where $\beta \in (H^p)^\perp$. Now $E^p(M) = (H^p)^\perp \oplus H^p$, we will show that $(H^p)^\perp = \Delta(E^p)$. We have $\Delta(E^p) \subset (H^p)^\perp$. For $\omega \in E^p$ and $\alpha \in H^p$,

$$
\langle \Delta \omega, \alpha \rangle = \langle \omega, \Delta \alpha \rangle = 0.
$$

On the other hand, we want to show $(H^p)^\perp \subset \Delta(E^p)$. To do this, we first prove the fact that there is a constant $c$ such that

$$
|\beta| \leq |\Delta \beta|, \forall \beta \in (H^p)^\perp.
$$
If this were not true, we would have a sequence \( \beta_j \in (H^p)^\perp \) with \( |\beta_j| = 1 \) and \( |\Delta \beta_j| \to 0. \), then by previous theorem, there is a subsequence of \( \beta_j \), wlog suppose it’s \( \{\beta_j\} \) itself, is Cauchy. \( \lim_{j \to \infty} \beta_j \) exist for every \( \psi \in E^p \). We define the linear functional on \( E^p \)

\[
l(\psi) = \lim_{j \to \infty} <\beta_j, \psi>
\]

Note that \( l \) is bounded and \( l(\Delta \varphi) = \lim_{j \to \infty} <\beta_j, \Delta \varphi> = <\Delta \beta_j, \varphi> = 0. \) So \( l \) is the weak solution of \( \Delta = 0. \) By previous theorem, there exists \( \beta \in E^p(M) \) such that \( l(\psi) = <\beta, \psi> \). And \( \beta_j \to \beta. \) Because \( |\beta_j| = 1 \) and \( \beta_j \in (H^p)^\perp \Rightarrow |\beta| = 1, \beta \in (H^p)^\perp. \) But by previous theorem, \( \Delta \beta = 0 \) and \( \beta \in H^p \) which is a contradiction.

Now, we can use this to prove \( (H^p)^\perp \subset \Delta E^p \). Let \( \alpha \in (H^p)^\perp \), we redefine our \( l \) to be

\[
l(\Delta \varphi) = <\alpha, \varphi>, \forall \varphi \in E^p.
\]

It is easy to check \( l \) is a well defined bounded linear functional on \( \Delta(E^p) \). Let \( \varphi \in E^p(M) \) and \( \psi = \varphi - H(\varphi) \), here \( H(\varphi) \) is the harmonic part of \( \varphi. \) We obtain

\[
|l(\Delta \varphi)| = |l(\Delta \psi)| = |<\alpha, \psi>| \leq |\alpha| \cdot |\Delta \psi| = c|\alpha| |\Delta \varphi|
\]

By Hahn Banach theorem, \( l \) extends to a bounded linear functional on \( E^p(M) \). \( l \) is a weak solution of \( \Delta \omega = \alpha. \) Hence we get the inclusion. \( (H^p)^\perp = \Delta(E^p) \) completes our proof of Hodge’s decomposition theorem.

**Remark 9** (Hahn Banach) Let \( p \) be a finite convex functional defined on a linear space \( L \), and let \( L_0 \) be the subspace of \( L. \) Suppose that \( f_0 \) is a linear functional on \( L_0 \) satisfying the condition \( f_0(x) \leq p(x) \) on \( L_0. \) Then \( f_0 \) can be extended to a linear functional \( f(x) \leq p(x) \) on the whole space \( L. \)

**Definition 10** We can also define Green’s operator, \( G : E^p(M) \to (H^p)^\perp \) by setting \( G(\alpha) \) equal to the unique solution of \( \Delta \omega = \alpha - H(\alpha) \in (H^p)^\perp. \) One can check that \( G \) is a bounded self adjoint operator which takes bounded sequences into sequences with Cauchy subsequences. Also, \( G \) commutes with \( d, \delta, \) and \( \Delta. \)

We have in fact a stronger result:

**Proposition 11** For any linear operator which \( T \) such that \( T\Delta = \Delta T. \)

**Proof**

We let \( T : E^p(M) \to E^q(M) \) be our linear operator such that \( T\Delta = \Delta T. \) Denote \( \pi_{(H^p)^\perp} \) to be the projection map on to \( (H^p)^\perp. \) By definition, \( G = (\Delta|(H^p)^\perp)^{-1} \circ \pi_{(H^p)^\perp}. \) From hodge decomposition theorem along with identity \( T\Delta = \Delta T, \) we have \( T(H^p) \subset H^q \) and \( T((H^p)^\perp) \subset (H^q)^\perp. \) We can now get

\[
T \circ \pi_{(H^p)^\perp} = \pi_{(H^p)^\perp} \circ T \Rightarrow \\
T \circ (\Delta|(H^p)^\perp) = (\Delta|(H^p)^\perp) \circ T
\]

and finally,

\[
T \circ (\Delta|(H^p)^\perp)^{-1} = (\Delta|(H^p)^\perp)^{-1} \circ T.
\]

This would imply directly that \( G \) commutes with \( T. \)
**Definition 12** A sequence of vector spaces and maps \((V_i, d_i)\), where \(d_i : V_i \rightarrow V_{i+1}\), is called a complex if \(d_{i+1} \circ d_i = 0\). Consider the complex

\[
\begin{align*}
\frac{d_{p-2}}{} & : \Omega^{p-1}(M) \xrightarrow{d_{p-1}} \Omega^p(M) \xrightarrow{d_p} \Omega^{p+1}(M) \xrightarrow{d_{p+1}} \\
\end{align*}
\]

and define \(H^p_{dR}(M) = \frac{\ker(d_i)}{\im(d_{i-1})}\)

**Theorem 13** Each de Rham cohomology class on a compact oriented Riemannian manifold \(M\) contains a unique harmonic representative.

**proof**
For any \(p\)-form on \(M\), we have

\[
\alpha = d\delta G \alpha + \delta G \delta \alpha + H \alpha = d\delta G \alpha + \delta G d \alpha + H \alpha
\]

by commutativity of Green’s operator. If \(\alpha\) is a closed form ,

\[
\alpha = d\delta G \alpha + H \alpha.
\]

Here \(H \alpha\) is a harmonic \(p\)-form in the same deRham cohomology class as \(\alpha\). If two harmonic forms \(\alpha_1\) and \(\alpha_2\) differ by an exact form \(d \beta\), then

\[
0 = d \beta + (\alpha_1 - \alpha_2).
\]

Note that \(<d \beta, \alpha_1 - \alpha_2> <\beta, \delta \alpha_1 - \delta \alpha_2> = <\beta, 0> = 0\). We have that \(d \beta = 0\) and \(\alpha_1 = \alpha_2\). Hence ,we showed that there is a unique harmonic form in each de Rham cohomology class.

**Remark 14** Recall that a closed form \(\alpha\) is a differential form whose exterior derivative \(d \alpha\) is zero. A exact form \(\alpha\) is a differential form such that there exist a form \(\beta\), \(d \beta = \alpha\)

4 Reference
1. Frank Warner. Foundations of Differentiable manifolds and Lie Groups, Chapter 6
2. Online Lecture Notes (introductory hodge theory) by Georgy Ivanov .