MATH 580, Final Project
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The Cole-Hopf transform provides an interesting method is solving the viscous Burgers' equation and has also opened up other doors to solve other higher order PDEs through similar methodologies. This paper will provide some physical justification to the need to solve the viscous Burgers' equation and then introduce the method. It will then point the interested reader to an example where the Cole-Hopf transformation is used to solve a higher order PDE.

History

The viscous Burgers' equation was presented in 1940 and in 1950 Hopf and in 1951 Cole independently introduced the method that has come to be known as the Cole-Hopf transformation to solve the viscous Burgers' equation. There is much conjecture as to the origin of the viscous Burgers' equation as it had appeared in previously published literature though Burger made it famous by introducing it as a simple model of turbulence and presenting some results on a preliminary investigation of the properties of the equation. There is also conjecture regarding the origin of the Cole-Hopf transformation as Forsyth introduced a method and equation in 1906 in his work on differential equations that can be transformed into the viscous Burgers' equation and the Cole-Hopf transformation. He did not discuss the equation, its applications or the transformation in any detail. [1]

Physical Justification

In the field of fluid dynamics, there exists an interesting problem called Stokes' First Problem in which an infinite plate with a fluid on top of it is impulsively set into motion with a constant speed. This problem allows for the study of the propagation of a disturbance by viscous friction alone. Some assumptions are made, namely: ρ=constant (density), μ or ν = constant (viscosity), P= constant (pressure) (see [2] for more information on Stokes' First Problem including boundary conditions and initial conditions). The assumption is also made that the layers of fluid are set in motion by viscous friction only. By starting with the PDE for continuity and x-momentum and applying the assumptions, the solution reduces to

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

With $\nu > 0$ being the viscosity, u the velocity field, t is time, y is the direction normal to the plate and α an arbitrary constant. This is the viscous Burgers' equation. The effect of adding the viscosity term (as compared to the inviscid Burgers' equation is to decrease the amplitude of u in time and also to prevent multi-valued solutions from developing[3]. In the case of Stokes' First problem, α is 0 though there are other situations where it is >0. With α=0, there are simpler solutions to Stokes' First problem though it can be solved by making use of the Cole-Hopf transform. The Cole-Hopf transform provides an important mechanism to solve equations of this form.

Cole-Hopf Transform

Hopf introduces[4] the transformation by first rewriting Burgers' equation as follows:

$$u_t = \left(\mu u_x - \frac{u^2}{2}\right)_x \tag{1}$$

And then introduces the dependant variable $\varphi = \varphi(x,t)$ where

$$\varphi(x,t) = exp\left\{-\frac{1}{2\mu}\int u\,\partial x\right\} \tag{2}$$

Which is inverted to be

$$u(x,t) = -2\mu\frac{\varphi_x}{\varphi} \tag{3}$$

Which when we substitute (3) into (1) and with a little manipulation, we get

$$-2\mu\left(\frac{\varphi_t}{\varphi}\right)_x = -\left(2\mu^2\frac{\varphi_{xx}}{\varphi}\right)_x \tag{4}$$

Which after integrating with respect to x and then redefining φ as $\varphi \cdot exp\{-\int C\,dt\}$ where C(t) is the constant from the integration with respect to x, then (4) can be rewritten as

$$\varphi_t = \mu\varphi_{xx} \tag{5}$$

Which is the heat equation.

Hopf stated:

> If u solves (1) in an open rectangle R of the x, t-plane and if u, u_x, u_{xx} are continuous in R then there exists a positive function φ of the form (2) that solves the heat equation in R and for which $\varphi, \varphi_x, \varphi_{xx}, \varphi_{xxx}$ are continuous in R. One easily shows that, conversely, every positive solution φ of (5) with the mentioned properties goes… over into a solution of (1) of the described general type. Let us call a function u that solves (1) in an x, t-domain D a regular solution in D if u, u_x, u_{xx} (and consequently u_t) are continuous in D. [5]

The heat equation (5) can be solved by making use of boundary conditions and initial conditions either from the original problem with the transformation applied to the conditions or by solving the equations over an infinite domain assuming that a solution is known for u(x,t=0) = g(x) to then solve for all of t>0.

Hopf goes on to prove that $\varphi \in C^\infty(R), \forall\, t > 0$ and the same for u. Uniqueness can be proven by supposing that u(x,t) is a regular solution of (1) in 0 < t < T and satisfies

$$\int_0^x u(\varepsilon,t)d\varepsilon \to \int_0^a u_0(\varepsilon)d\varepsilon \ as\ x \to a, t \to 0$$

where

$$\int_0^x u_0(\varepsilon)d\varepsilon = o(x^2)\ for\ |x|\ large$$

and u_0 is integrable in every finite x-interval, then $u = -2\nu(\varphi_x/\varphi)$. By discretizing the space into strips and then looking at the sign of the limit as t approaches 0 by making use of Widder's theorem on non-negative solutions to the heat equation, Hopf shows that "$\varphi(x,t)$ is uniquely determined by the initial values… u(x,t) is, therefore, completely unique". [5]

By choosing $f(x) = -2\nu \frac{\partial_x \varphi(x,t)}{\varphi(x,t)}$ we can then solve the ODE to get [6]

$$\varphi(x,0) = h(x) = Ce^{-F(x)}$$

where

$$F(x) = \frac{1}{2\nu}\int_0^x f(s)ds.$$

We can then solve for $\varphi(x,t)$ to get

$$\varphi(x,t) = \frac{C}{\sqrt{4\pi\nu t}}\int_R h(s)e^{-\frac{(x-s)^2}{4\nu t}}ds$$

The solution to u(x,t) can then be determined from (3) above.

Further Information

Despite the simplicity of the Cole-Hopf transform, there are more effective ways to solve the viscous Burgers' equation. For example, Gorguis[7] suggests that the Adomian decomposition method is more effective to solve the viscous Burger's equation as it provides a solution with less effort. There are however other situations where the Cole-Hopf transformation allows for the solving of some very important equations in other problems. For example, it has been used to solve general seventh-order Korteweg de Vries equation given by:

$$u_t + au^3u_x + bu_x^3 + cuu_xu_{xx} + du^2u_{xxx} + eu_{2x}u_{3x} + fu_xu_{4x} + guu_{5x} + u_{7x} = 0$$

which is introduced as equation 1.1 in [8] which is used to discuss the stability of the "Korteweg de Vries equations under a singular perturbation". The Cole-Hopf transformation has also been generalized by taking a variable coefficient in the transform so as to derive both "non-linear parabolic and hyperbolic equations which are exactly linearisable."[9] These equations include the turbulence model proposed by Burgers and a reaction-diffusion equation from Goldstein-Murray and the interested reader is encouraged to review reference [9].

References

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