Definition 1. *Let $f \in \mathcal{C}^\infty(\mathbb{C}, \mathbb{C})$. For $z = x + iy \in \mathbb{C}$, write $f(x, y) = u(x, y) + iv(x, y)$, for $u, v \in \mathcal{C}^\infty(\mathbb{C}, \mathbb{R})$. Then the* pushforward *of f at z is the linear map f_* between $T_z\mathbb{C}$ and $T_{f(z)}\mathbb{C}$ given in the natural coordinates by the matrix*

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Note that since $\mathbb{C}$ is given by one chart, we identify all of the tangent spaces with $\mathbb{R}^2$, so we only care about the matrix.

Example 1. *Let $f : \mathbb{C} \to \mathbb{C} : z \mapsto iz$. Then*

$$f_* = [i]_\mathbb{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $[\alpha]_\mathbb{R}$ denotes the real matrix corresponding to multiplication by α in the real coordinates, for each $\alpha \in \mathbb{C}$.

We have just described the pushforward of multiplication by i as a real matrix using the real coordinates on $\mathbb{C}$. We will see that this matrix is all that we need in order to do complex analysis, and hence we call it the *complex structure* of $\mathbb{C}$, and denote it by J. Note that J is simply counter-clockwise rotation by $\frac{\pi}{2}$.

Definition 2. *Recall that f as above is* holomorphic *if it satisfies the Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Note that these equations are equivalent to $Jf_* = f_*J$. We will take this commutation relation as the definition of f being holomorphic.

Example 2. *For $\alpha \in \mathbb{C}$, let $f : \mathbb{C} \to \mathbb{C} : z \mapsto \alpha z$. Then $f_* = [\alpha]_\mathbb{R}$ is simply rotation by $\arg \alpha$ and dilation by $|\alpha|$, and both actions commute with the rotation of J so that f is holomorphic.*

Example 3. *For $n \in \mathbb{N}$, let $f : \mathbb{C} \to \mathbb{C} : z \mapsto z^n$. Then $f_*(z) = [nz^{n-1}]_\mathbb{R}$ is again multiplication by a complex number (varying this time on the base point z), so is holomorphic.*

Example 4. *Let $f : \mathbb{C} \to \mathbb{C} : z \mapsto \bar{z}$. Then f_* is a flip along the real axis, which does not commute with the rotation J, and hence is not holomorphic.*

Definition 3. *For M a manifold, $J \in \Gamma(\mathrm{End}(TM))$ is a* complex structure *for M if $J^2 = -\mathrm{Id}$ and it satisfies some more technical condition. We will see some examples which may give intuition on this. We say that (M, J) is a complex manifold.*

Definition 4. *Let* (M, J) *and* (N, K) *be complex manifolds. A function* $f : M \to N$ *is* holomorphic *if* $f_* J = K f_*$.

Example 5. *For* J *the standard complex structure on* $\mathbb{C}$, $-J$ *is also a complex structure corresponding to multiplication by* $-i$. *Clearly a function is holomorphic with respect to* $-J$ *if and only if it is holomorphic with respect to* J.

Example 6. *Complex structures on* $\mathbb{R}^4$.

Three different complex structures on $\mathbb{R}^4$ are:

$$I = \begin{pmatrix} \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}, J = \begin{pmatrix} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}, K = \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}.$$

These are generated by decomposing $\mathbb{R}^4$ into $\mathbb{R}^2 \oplus \mathbb{R}^2$, and then putting complex structures on each $\mathbb{R}^2$ to agree with eachother's orientation. For example, the matrix

$$\tilde{J} = \begin{pmatrix} \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \end{pmatrix}$$

satisfies $\tilde{J}^2 = -\operatorname{Id}$, but the rotation defined does not agree with the natural orientation of $\mathbb{R}^4$, and it is not a complex structure. There are also similar "almost" complex structures $\tilde{I}$ and $\tilde{K}$ defined analogously.

Example 7. *(Infinitesimal) Isometries on* $(\mathbb{R}^4, < \cdot, \cdot >)$.

The isometries on Euclidean 4-space is the group of Euclidean motions, $\mathbb{R}^4 \rtimes O_4(\mathbb{R})$, where the $\mathbb{R}^4$ part are the translations, and O^4 are the rotations. We call the corresponding Lie algebra $\mathbb{R}^4 \oplus \mathfrak{so}_4(\mathbb{R})$ the *infinitesimal isometries* of Euclidean 4-space. We have already described six different rotations: $I, J, K, \tilde{I}, \tilde{J}, \tilde{K}$. It is easy to see that the Lie algebra elements in $\mathfrak{so}_4(\mathbb{R})$ generating these are linearly independent. Since $\mathfrak{so}_4(\mathbb{R})$ is six dimensional, these generate all of them. Infinitesimal isometries are *holomorphic* if the maps that they generate are holomorphic. The translations are clearly holomorphic with respect to any complex structure, since they act trivially on the tangent space. It turns out that $\tilde{I}, \tilde{J}$, and $\tilde{K}$ commute with all of the complex structures, but that I, J, and K do not commute with eachother. In conclusion, for a fixed complex structure, all of the infinitesimal isometries are holomorphic, except for a two dimensional subspace corresponding to the other two complex structures. This contrasts with the following theorem in the compact case:

Theorem 1. *(Lichnerowitz)On a compact Kähler manifold, every infinitesimal isometry is holomorphic.*

Note that a *Kähler manifold* is complex manifold with a metric that behaves similarly to the way that the standard inner product $< \cdot, \cdot >$ behaves on $\mathbb{C}^n$. We have seen that this theorem is false when relaxing the compactness condition. I have done some research looking into relaxing the Kähler condition. Interestingly, the known counterexamples come from an interplay between multiple complex structures analogous to what I've described for $\mathbb{R}^4$.