THE CARDY-SMIRNOV FORMULA

1. Description of the Problem

Let T be a planar lattice whose faces are equilateral triangles. Consider a site percolation where each site of T is declared to be open (or black colored) with probability p and closed (or white colored) with probability $1-p$. When dealing with triangular lattice, it is often convenient to associate to each site A of T a hexagon Λ_A so that the tiling of the plane by hexagons is obtained. Thus, a site percolation on T is equivalent to a face percolation on hexagonal (honeycomb) lattice Λ. H. Kesten and J. C. Wierman showed that:
i. If $p \leq 1/2$, then there is no infinite connected component of open sites almost surely.
ii. If $p > 1/2$, then there is a unique infinite connected component of open sites almost surely.
This threshold value is called critical probability. Throughout this note p is set be equal to $1/2$. Let δT denote triangular lattice obtained by scaling T by a factor δ and let $D_4 = (D; a, b, c, d)$ be a simply connected, bounded domain with boundary points a, b, c, d appearing in anticlockwise order. We are interested in the event that D_4 has an open crossing, i.e., a path consisting of open sites, from da to bc on the lattice δT. It was conjectured by M. Aizenman that the limit of the above probability exists as $\delta \to 0$ and is invariant under conformal map ϕ. Let us write $\mathbf{P}^{\delta}_{1/2}(D_4, T) = \mathbf{P}(D_4$ has an open crossing from da to bc on the lattice $\delta T)$. In 1992, J. Cardy predicted that the crossing probability is equal to

$$\mathbf{P}(D_4) = \lim_{\delta\to 0} \mathbf{P}^{\delta}_{1/2}(D_4, T) = \tfrac{3\Gamma(2/3)}{\Gamma(1/3)^2}\varphi^{1/3} F_1(1/3, 2/3; 4/3; \varphi),$$

where $\varphi = \frac{(d-c)(b-a)}{(d-b)(c-a)}$ and ${}_2F_1(r, q, s; t) = \sum_{n=0}^{\infty} \frac{r^{(n)}q^{(n)}}{s^{(n)}} \frac{t^n}{n!}$ with $x^{(n)} = x(x+1)(x+2)\ldots(x+n-1)$, but he didn't give a rigorous foundation to the formula. Later L. Carleson made an important observation that, in the case of equilateral triangle, Cardy's prediction can be rephrased in a very simple form: if D_4 is conformally mapped into a unit triangle (an equilateral triangle with side-length one) $A_1A_\tau A_{\tau^2}$, i.e., $\phi(a) = A_1$, $\phi(b) = A_\tau$, $\phi(c) = A_{\tau^2}$, $\phi(d) = X \in A_1A_{\tau^2}$ where $\tau = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, then the probability that there exists an open crossing from da to bc on δT within $A_1A_\tau A_{\tau^2}$ is equal to $|XA_1|$. The conjecture remained open until Smirnov (2001) emerged with an astonishingly elegant proof. The objective here is to outline Smirnov's proof of Cardy's formula in Carleson's form.

2. Main Result

Theorem 1 (Smirnov 2001) *Cardy's prediction is true.*

Proof

We shall use some preliminary lemmas without proof.

Lemma 1 (Russo-Seymour-Welsh) *Let $A_z(r, R)$ denote the annulus centered at z with inner radius r and outer radius R. If $R \geq 2r$ and $r \geq 1000\delta$, then the probability that $A(r, R)$ has an open crossing from a site inside $C_r(z)$ to a site outside $C_R(z)$ on δT is at most $(R/r)^{\beta}$, where β is some universal constant.*

Let z be a center of a triangle in δT and $E_1^\delta(z)$ denote the event that there exists a black (i.e., open) path from A_1A_τ to $A_{\tau^2}A_1$ separating z from $A_\tau A_{\tau^2}$. Likewise, define
$E_\tau^\delta(z)$ = there exists a black path from $A_\tau A_{\tau^2}$ to A_1A_τ separating z from $A_{\tau^2}A_1$,
$E_{\tau^2}^\delta(z)$ =there exists a black path from $A_{\tau^2}A_1$ to $A_\tau A_{\tau^2}$ separating z from A_1A_τ.

Lemma 2 (Color Switching) *Let $s_1s_2s_3$ be a triangular face in $A_1A_\tau A_{\tau^2}$ and z be its center (a site in the dual $\delta\Lambda$ lattice). Also, let z_1, z_2, z_3 denote the centers of the three adjacent faces with the same orientation as $A_1A_\tau A_{\tau^2}$ such that s_j is the vertex opposite to z_j. Then the event $E_1^\delta(z_1) \setminus E_1^\delta(z)$ is equivalent to the event that there exist disjoint paths such that i) black paths from s_2 and s_3 to $A_1A_{\tau^2}$ and A_1A_τ respectively, and ii) a white path from s_1 to $A_\tau A_{\tau^2}$.*

Define $H_1^\delta(z) = \mathbf{P}[E_1^\delta(z)]$, $H_\tau^\delta(z) = \mathbf{P}[E_\tau^\delta(z)]$ and $H_{\tau^2}^\delta(z) = \mathbf{P}[E_{\tau^2}^\delta(z)]$. Ultimately we want to show that these functions are harmonic in $A_1A_\tau A_{\tau^2}$ and then characterize them by their boundary conditions. The following lemma follows easily from Lemma 2 and is of great importance in evaluating the discrete contour integral that we discuss below.

Lemma 3 (Separating Probability) *The following relations hold:*

$$\mathbf{P}[E_1^\delta(z_1) \setminus E_1^\delta(z)] = \mathbf{P}[E_\tau^\delta(z_2) \setminus E_\tau^\delta(z)] = \mathbf{P}[E_{\tau^2}^\delta(z_3) \setminus E_{\tau^2}^\delta(z)].$$

Define $h_1^\delta(w,z) = \mathbf{P}[E_1^\delta(z) \setminus E_1^\delta(w)]$, $h_\tau^\delta(w,z) = \mathbf{P}[E_\tau^\delta(z) \setminus E_\tau^\delta(w)]$ and $h_{\tau^2}^\delta(w,z) = \mathbf{P}[E_{\tau^2}^\delta(z) \setminus E_{\tau^2}^\delta(w)]$. It is straightforward to see that $H_1^\delta(z) - H_1^\delta(w) = h_1^\delta(w,z) - h_1^\delta(z,w)$, $H_\tau^\delta(z) - H_\tau^\delta(w) = h_\tau^\delta(w,z) - h_\tau^\delta(z,w)$, and $H_{\tau^2}^\delta(z) - H_{\tau^2}^\delta(w) = h_{\tau^2}^\delta(w,z) - h_{\tau^2}^\delta(z,w)$ hold. Note that any triangular contour inside $A_1A_\tau A_{\tau^2}$ can be approximated sufficiently well by union of discrete triangular contour Γ with anticlockwise orientation i.e., equilateral triangle whose corners are sites of δT and whose sides are parallel to bonds of δT. Then the discrete contour integral, $\oint_{\partial\Gamma} H_j^\delta(z)dz$ for $j = 1, \tau, \tau^2$, is defined as the usual contour integral for a piecewise constant function $H_j^\delta(z)$; for any $z \in \partial\Gamma$, $H_j^\delta(z)$ is evaluated at the nearest vertex of $\delta\Lambda$ (the center of the triangular face) inside Γ . For example, if $\partial\Gamma$ is a triangle with side-length δ and center w, then $H_j^\delta(z) = H_j^\delta(w)$ for all $z \in \partial\Gamma$.

Lemma 4 (Discrete Contour Integral) *The following relations hold:*

$$\oint_{\partial\Gamma} H_1^\delta(z)dz = \oint_{\partial\Gamma} \frac{H_\tau^\delta(z)}{\tau}dz + O(\delta^\beta) = \oint_{\partial\Gamma} \frac{H_{\tau^2}^\delta(z)}{\tau^2}dz + O(\delta^\beta)$$

for some constant β.

Proof of Lemma 4

It suffices to prove the first equality. Without loss of generality, we may assume that z is at distance at least $a > 2000\delta$ from A_1A_τ. If $E_j^\delta(w) \setminus E_j^\delta(z)$ holds for some $j = 1, \tau, \tau^2$, then applying Lemma 2 one can conclude that the monochromatic (entirely open or entirely closed) paths from the three vertices s_1, s_2, s_3 will constitute open or closed crossings of the annulus $A_z(\delta, a)$. Since

this will also form an open crossing in $A_z(1000\delta, a)$, applying Lemma 1 with $r = 1000\delta$ and $R = a$ we deduce that the probability of such a crossing is at most $2(1000\delta/a)^\beta = O(\delta^\beta)$. Thus, $h_j^\delta(z,w) = O(\delta^\beta)$ uniformly over z, w with z being the center of a triangular face in Γ and $z \sim w$, where $\sim$ denotes adjacency in $\delta\Lambda$. Let

$$S_j = \sum_{z\in\Gamma^0}\sum_{w\sim z}(w-z)h_j^\delta(z,w),$$

where Γ^0 is the set of sites of $\delta\Delta$ within Γ, and w runs over the three neighbors z_1, z_2, z_3 (in anticlockwise order) of z in $\delta\Lambda$. Obviously $z_1 - z = \frac{1}{\tau}(z_2 - z) = \frac{1}{\tau^2}(z_3 - z)$. From Lemma 3 we have $h_1^\delta(z,z_1) = h_\tau^\delta(z,z_2) = h_{\tau^2}^\delta(z,z_3)$. Furthermore, permuting the subscripts of z gives

$$h_1^\delta(z,z_2) = h_\tau^\delta(z,z_3) = h_{\tau^2}^\delta(z,z_1)$$

and

$$h_1^\delta(z,z_3) = h_\tau^\delta(z,z_1) = h_{\tau^2}^\delta(z,z_2).$$

Setting $z_4 = z_1$ and using the above relations we see

$$\begin{aligned}\sum_{k=1}^{3}(z_k - z)h_\tau^\delta(z,z_k) &= \sum_{k=1}^{3}(z_{k+1} - z)h_\tau^\delta(z,z_{k+1})\\ &= \sum_{k=1}^{3}\tau(z_k - z)h_\tau^\delta(z,z_{k+1})\\ &= \tau\sum_{k=1}^{3}(z_k - z)h_1^\delta(z,z_k).\end{aligned}$$

Summing over $z \in \Gamma^o$ yields $S_\tau = \tau S_1$. Next we break up S_j into two parts as follows:

$$\begin{aligned}S_j &= \sum_{z\sim w, z\in\Gamma^0}(w-z)h_j^\delta(z,w)\\ &= U_j + V_j,\end{aligned}$$

where $U_j = \sum_{z\sim w, z, w\in\Gamma^0}(w-z)h_j^\delta(z,w) + (z-w)h_j^\delta(w,z)$ and $V_j = \sum_{z\sim w, z\in\Gamma^0, w\notin\Gamma^0}(z-w)h_j^\delta(w,z)$. Note that $|w-z| < 0$ implies $|(w-z)h_j^\delta(z,w)| \le \delta\sup_{z\sim w} h_j^\delta(z,w)$. As V_j has $O(L/\delta)$ terms, it follows that

$$V_j = \sum_{z\in w, z\in\Gamma^0, w\notin\Gamma^0}(w-z)h_j^\delta(z,w) = O(L(\delta/a)^\beta)$$

for each $j = 1, \tau, \tau^2$. For any adjacent pair $z, w \in \Gamma^0$, we have

$$\begin{aligned}(w-z)h_j^\delta(z,w) + (z-w)h_j^\delta(w,z) &= (w-z)(h_j^\delta(z,w) - h_j^\delta(w,z))\\ &= (w-z)(H_j^\delta(w) - H_j^\delta(z).\end{aligned}$$

Summing over all unordered pairs z, w with $z \sim w$ and $z, w \in \Gamma^0$ gives

$$\begin{aligned} U_j &= \sum_{z\in\Gamma^0} \sum_{w\sim z, w\in\Gamma^0} (z-w)H_j^\delta(z) \\ &= \sum_{z\in\Gamma^0}\sum_{w\sim z}(z-w)H_j^\delta(z) - \sum_{w\in\Gamma^0}\sum_{w\sim z, z\notin\Gamma^0}(z-w)H_j^\delta(z) \\ &= \sum_{w\in\Gamma^0}\sum_{w\sim z, z\notin\Gamma^0}(w-z)H_j^\delta(z). \end{aligned}$$

When z is a center of a triangular face meeting $\partial\Gamma$ and $\overrightarrow{zw}$ crosses $\partial\Gamma$ from the inside to the oustide, the dual bond $\overrightarrow{xy} \in \partial\Gamma$ of δT is obtained first by rotating $\overrightarrow{zw}$ clockwise by $\pi/2$ and then scaling by $1/\sqrt{3}$. Thus, $w - z = (-i/\sqrt{3})(y - x)$. Recalling the definition of discrete contour integral we have

$$U_j = \sum_{z\in\Gamma^0}\sum_{w\sim z, w\notin\Gamma^0} = -\frac{i}{\sqrt{3}}\oint_{\partial\Gamma} H_j^\delta(z)dz.$$

Using this in conjuction with $S_\tau = \tau S_1 = U_\tau + V_\tau = \tau U_1 + \tau V_1$ gives

$$U_\tau - \tau U_1 = -\frac{i}{\sqrt{3}}\oint_{\partial\Gamma} H_\tau^\delta(z)dz + \tau\frac{i}{\sqrt{3}}\oint_{\partial\Gamma} H_1^\delta(z)dz = -V_\tau + \tau V_1.$$

Also, we have $|U_\tau - \tau U_1| = \frac{1}{\sqrt{3}}|\oint_{\partial\Gamma} H_\tau^\delta(z)dz - \tau\oint_{\partial\Gamma} H_1^\delta(z)dz| = |V_\tau - \tau V_1| \le |V_\tau| + |V_1| = O(L(\delta/a)^\beta)$. Hence it follows that

$$\oint_{\partial\Gamma} H_1^\delta(z)dz = \oint_{\partial\Gamma}\frac{H_\tau^\delta(z)}{\tau}dz + O(\delta^\beta)$$

thereby completing the proof of Lemma 4.
We can see that H_j^δ continuously extends to the closure of $A_1A_\tau A_{\tau^2}$ as follows. Recall that H_j^δ is defined at the centers of triangular faces. At the center z of a trianglular face meeting the boundary, set $H_j^\delta = H_j^\delta(w)$ where w is the closest point w at which H_j^δ is defined. If w is not unique, choose one arbitrarily. Further, at the corner s of a triangular face meeting the boundary of $A_1A_\tau A_{\tau^2}$, set H_j^δ to be the average of the values at the center of all triangles incident with s and meeting the boundary. Next, divide each face of δT into three triangles by joining the vertices with its center, and define H_j^δ on each of these triangles by linear interpolation.
Using Lemma 1 one can show that $H_j^\delta(z)$ is uniformly equicontinuous. Since these functions are uniformly bounded, invoking the Arzelà -Ascoli Theorem shows that $(H_1^\delta, H_\tau^\delta, H_{\tau^2}^\delta)$ contains uniformly convergent subsequence with a continuous limiting functions $(H_1, H_\tau, H_{\tau^2})$. Thus, letting $\delta \to 0$ in the discrete contour integrals we see that for any equilateral triangle $\partial\Gamma$,

$$\oint_{\partial\Gamma} H_1(z)dz = \oint_{\partial\Gamma}\frac{H_\tau(z)}{\tau}dz = \oint_{\partial\Gamma}\frac{H_{\tau^2}(z)}{\tau^2}dz$$

where the integral is the usual contour integral. Define

$$G_1 = H_1 + H_\tau + H_{\tau^2} \text{ and } G_2 = H_1 + \tau H_\tau + \tau^2 H_{\tau^2}.$$

Applying Lemma 4 yields $\oint_{\partial\Gamma}(H_1 + H_\tau + H_{\tau^2})dz = (1+\tau+\tau^2)\oint_{\partial\Gamma} H_1 = 0$ and $\oint_{\partial\Gamma}(H_1 + \tau H_\tau + \tau^2 H_{\tau^2})dz = (1+\tau+\tau^2)\oint_{\partial\Gamma} H_1 dz = 0$. By Morera's theorem these two functions are holomorphic in $A_1A_\tau A_{\tau^2}$. Since the function $G_1(z)$ is both holomorphic and real valued, it must be constant. Then noting that $H_{\tau^2}(z) = 0$ and $H_1(z) + H_\tau(z) = 1$ for all $z \in A_1A_\tau$, we obtain $G_1(z) = 1$ for all $z \in A_1A_\tau A_\tau^2$. Therefore,

$$H_1 = \frac{2H_1 - H_\tau - H_{\tau^2}}{3} + \frac{H_1 + H_\tau + H_{\tau^2}}{3} = \frac{2\Re(G_2)}{3} + \frac{1}{3}.$$

Similarly we obtain

$$H_\tau = \frac{2\Re(G_2/\tau)}{3} + \frac{1}{3} \text{ and } H_{\tau^2} = \frac{2\Re(G_2/\tau^2)}{3} + \frac{1}{3}.$$

Since it is the real part of holomorphic function, the $H_j, j = 1, \tau, \tau^2$ is harmonic. Recall that the directional derivative of complex-valued function G is defined as

$$\frac{\partial G}{\partial \eta}(z) = \lim_{h\to 0, h\in\mathbb{R}} \frac{G(z+\eta h) - G(z)}{h}$$

for each η with $|\eta| = 1$. Obviously, if G is holomorphic, it must hold that $\frac{\partial G}{\partial 1} = \frac{1}{\tau}\frac{\partial G}{\partial \tau} = \frac{1}{\tau^2}\frac{\partial G}{\partial \tau^2}$. Taking the real parts we get $\frac{\partial \Re(G)}{\partial 1} = \frac{\partial \Re(G/\tau)}{\partial \tau} = \frac{\partial \Re(G/\tau^2)}{\partial \tau^2}$.
Now we determine the boundary conditions. Note that $H_{\tau^2} = 0$ on $A_\tau A_{\tau^2}$ and $H_{\tau^2}(A_{\tau^2}) = 1$. Since $H_1 = 0$ on $A_\tau A_{\tau^2}$, $\frac{\partial H_{\tau^2}}{\partial 1} = \frac{\partial H_1}{\partial \tau} = 0$ on $A_\tau A_{\tau^2}$. Similar argument gives $\frac{\partial H_{\tau^2}}{\partial 1} = 0$ on $A_{\tau^2}A_1$, so the horizontal derivative of H_{τ^2} vanishes on $A_1A_{\tau^2} \cup A_\tau A_{\tau^2}$. The only function that satisfies these requirement is the normalized height function given by $H_{\tau^2}(z) = \frac{2}{\sqrt{3}}\text{dist}(z, A_1A_\tau)$. Finally, when $z = X \in A_{\tau^2}A_1$, the theorem follows. ■

General Domain

For a general simply connected, bounded domain D_4, the proof proceeds almost the same way as above although more care should be taken when handling the boundaries. For example, one needs to show that approximating D_4 by a discrete domain (a component of honeycomb lattice) doesn't alter the crossing probability significantly. If the triplet $(H_1, H_\tau, H_{\tau^2})$ is the separating probabilities for D_4, then $(H_1 \circ \phi^{-1}, H_\tau \circ \phi^{-1}, H_{\tau^2} \circ \phi^{-1})$ corresponds to the probabilities for the unit triangle $A_1A_\tau A_{\tau^2}$. In fact, it can be shown that the function $G_2 = H_1 + \tau H_\tau + \tau^2 H_{\tau^2}$ coincides with the conformal map ϕ sending D_4 to $A_1A_\tau A_{\tau^2}$.

3. Remark

An extremely detailed explanation of Smirnov's proof might be found in [1]. [2] contains many beautiful diagrams. [4] is a great source on two-dimensional percolation and its connection to Schramm-Loewner Evolution.

Bibliography

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