MATH 566, Final Project
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The *unrestricted partition function* $p(n)$ counts the number of ways a positive integer n can be expressed as a sum of positive integers $\leq n$.
For example: $p(4) = 5$, since 4 can be written as $1+1+1+1,\ \ 1+1+2,\ \ 2+2,\ \ 3+1,\ \ 4$.

This project intends to present a proof of the following remarkable result:

Theorem: If $n \geq 1$, then

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left[\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right] \tag{1}$$

$$\text{where} \quad A_k(n) = \sum_{0\leq h<k} \exp\left(\pi i s(h,k) - 2\pi i n\frac{h}{k}\right) \qquad \text{and} \qquad s(h,k) = \sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right) \tag{2}$$

This theorem, proved by Rademacher in 1937 [6], greatly improves the asymptotic relation discovered by both Hardy and Ramanujan in 1918 [4] and J.V.Uspensky in 1920 [7] independently, and which stated that:

$$p(n) \sim \frac{1}{4n\sqrt{3}}e^{K\sqrt{n}} \quad \text{as } n\to\infty \qquad \text{where } K = \pi\sqrt{\frac{2}{3}} \tag{3}$$

The proof of this formula involves the *circle method*, a tool used many times by Hardy, Littlewood and Ramanujan in tackling asymptotic problems of additive number theory. For example, this method provides a lot of insight for a possible proof of Goldbach's conjecture, which states that every integer is the sum of three primes [5]. Later on, Erdős proved that this formula could be derived by elementary means [3]. Hardy and Ramanujan also proved the following exact asymptotic formula:

$$p(n) = \sum_{k<\alpha\sqrt{n}} P_k(n) + \mathscr{O}(n^{-1/4}) \tag{4}$$

where α is a constant, and $P_1(n)$ is the dominant term , asymptotic to $\frac{1}{4n\sqrt{3}}e^{K\sqrt{n}}$. Note that the infinite sum $\sum_{k=1}^{+\infty} P_k(n)$ diverges for each n. However, Rademacher's choice of contour of integration, which also uses the circle method, yields a formula involving a convergent series.

The following proof is taken from [2], and is divided into 6 main parts.

Proof. 1. Generating function corresponding to $p(n)$:

We shall make use of the following lemma:

Lemma: If $|x| < 1$, then $\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}\frac{1}{1-x^k} =: F(x)$.

Proof. This proof can be found in [1]. Restrict x to lie on the interval $0 \leq x < 1$ and introduce two functions

$$F_m(x) = \prod_{k=1}^{m}\frac{1}{1-x^k} \qquad \text{and} \qquad F(x) = \prod_{k=1}^{\infty}\frac{1}{1-x^k} = \lim_{m\to\infty} F_m(x) \tag{5}$$

Since for $0 \leq x < 1$, $\sum x^k$ converges absolutely, so does $\prod(1-x^k)$, and hence $F(x)$. Also

$$F_{m+1}(x) = \prod_{k=1}^{m+1}\frac{1}{1-x^k} = \frac{1}{1-x^{m+1}}\prod_{k=1}^{m}\frac{1}{1-x^k} = \frac{1}{1-x^{m+1}}F_m(x) \geq F_m(x) \tag{6}$$

so that for fixed x, $F_m(x) \le F(x)$ for every m. Now,

$$F_m(x) = \prod_{k=1}^{m} \frac{1}{1-x^k} = \prod_{k=1}^{m} \left(\sum_{n=1}^{\infty} (x^k)^n \right) = 1 + \sum_{k=1}^{\infty} p_m(k) x^k \quad \Longrightarrow \text{ absolutely convergent since each } \sum_{n=1}^{\infty} (x^k)^n \text{ is.}$$

where $p_m(k)$ is the number of solutions of the equation $k = k_1 + 2k_2 + \ldots + mk_m$, ie: the number of partitions of k into parts not exceeding m. Since we have $p_m(k) \le p(k)$, with $p_m(k) = p(k)$ if $m \ge k$, it holds that $\lim_{m\to\infty} p_m(k) = p(k)$. Split $F_m(x)$ into two parts:

$$F_m(x) = 1 + \sum_{k=1}^{\infty} p_m(k)x^k = \sum_{k=0}^{m} p_m(k)x^k + \overbrace{\sum_{k=m+1}^{\infty} p_m(k)x^k}^{>0} \quad \Longrightarrow \quad \sum_{k=0}^{m} p(k)x^k \le F_m(x) \le F(x) \tag{7}$$

Therefore $\sum_{k=0}^{\infty} p(k)x^k$ converges. Also

$$\sum_{k=0}^{\infty} p_m(k)x^k \le \sum_{k=0}^{\infty} p(k)x^k \le F(x) \tag{8}$$

so for each fixed x, $\sum p_m(k)x^k$ converges uniformly in m. Taking $\lim_{m\to\infty}$:

$$F(x) = \lim_{m\to\infty} F_m(x) = \lim_{m\to\infty} \sum_{k=0}^{\infty} p_m(k)x^k = \sum_{k=0}^{\infty} \lim_{m\to\infty} p_m(k)x^k = \sum_{k=0}^{\infty} p(k)x^k \tag{9}$$

This proves the lemma for $0 \le x < 1$. Take the analytic continuation to the unit disk $|x| < 1$. □

$F(x)$ is absolutely convergent on $|x| < 1$. Dividing on both sides of the above by x^{n+1} yields the *Laurent series expansion of* $\frac{F(x)}{x^{n+1}}$, valid for $0 < |x| < 1$:

$$\frac{F(x)}{x^{n+1}} = \sum_{k=0}^{\infty} \frac{p(k)x^k}{x^{n+1}} \tag{10}$$

This expansion has a simple pole at $x = 0$. Therefore, by Cauchy Residue Theorem,:

$$\mathrm{Res}_{x=0} \frac{F(x)}{x^{n+1}} = p(n) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{F(x)}{x^{n+1}} dx \tag{11}$$

$\mathscr{C}$ being any contour in $0 < |x| < 1$ enclosing 0.

We now move from the x-plane to the τ-plane by performing the following change of variables: $x = e^{2\pi i \tau}$. The punctured disk is mapped to the infinite strip $0 < \mathrm{Re}(x) < 1$ in the τ-plane, and in particular, the circle $\gamma : [0,1] \to \mathbb{D}^{\times}(0)$ given by $\gamma(a) = e^{-2\pi + ia}$ is mapped to the straight line joining i to $i+1$. Integrating along γ in this plane gives:

$$p(n) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(e^{2\pi i \tau})}{e^{2\pi i \tau n}} d\tau \tag{12}$$

2. Rademacher's path:

We now wish to replace the path of integration γ by a path that lies close to the singularities of $F(x)$. Since $1 - x^k = 0$ for $x = e^{2\pi i h/k}$, where $(h,k) = 1$ with $0 \le h < k$, we see that we must take a path lying near all roots of unity. But before defining it, two definitions are in order:

- *Farey fractions:* Def: The Farey fractions of order n are defined as the set of reduced fractions in the close interval $[0,1]$ with denominators $\le n$, listed in increasing order of magnitude. For example:

$$F_1: \quad \frac{0}{1}, \frac{1}{1} \qquad F_2: \quad \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \qquad F_3: \quad \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \qquad \ldots \tag{13}$$

- *Ford circles:* Def: The Ford circle $\mathscr{C}(h,k)$, where $(h,k)=1$ with $0 \le h < k$, is a circle of radius $\frac{1}{2k^2}$, with center located at $\frac{h}{k} + \frac{i}{2k^2}$.
  Note that $\mathscr{C}(h,k)$ is tangent to the real axis at the point $x = \frac{h}{k}$.

The following lemma is given without proof (the proof isn't hard, but it is lenghty):

Lemma: Ford circles of consecutive Farey fractions are tangent to each other. If $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are consecutive in F_N, the points of tangency are given by:

$$\alpha_1(h,k) = \frac{h}{k} - \frac{k_1}{k(k^2+k_1^2)} + \frac{i}{k^2+k_1^2} \qquad \text{and} \qquad \alpha_2(h,k) = \frac{h}{k} - \frac{k_2}{k(k^2+k_2^2)} + \frac{i}{k^2+k_2^2} \tag{14}$$

So for example, the five Ford circles associated to the Farey fractions $F_3 = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$ are tangent to each other, as illustrated on Figure 1.

With these facts at hand, it is now possible to define Rademacher's path $P(N)$ in the following way:

Def: Consider the Ford circles of the Farey series F_N. If $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are consecutive in F_N, the points of tangency of $\mathscr{C}(h_1,k_1)$, $\mathscr{C}(h,k)$ and $\mathscr{C}(h_2,k_2)$ divide $\mathscr{C}(h,k)$ into two arcs, an upper arc and a lower arc. $P(N)$ is the union of the upper arcs so obtained.
See Figure 1 for an illustration of $P(3)$.

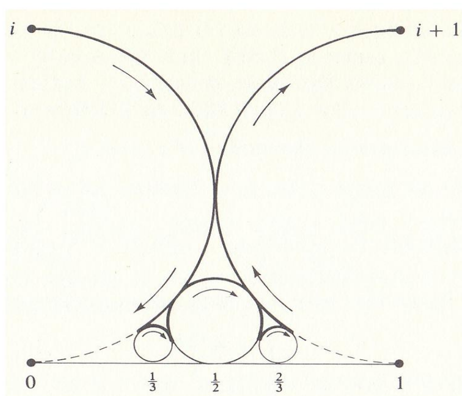

Figure 1:

For the moment we keep N fixed, and will let it go to $+\infty$ only near the end of the proof. We may then rewrite the path of integration as:

$$\int_i^{i+1} = \int_{P(N)} = \sum_{k=1}^{N} \sum_{0\le h<k,\ (h,k)=1} \int_{\gamma(h,k)} =: \sum_{h,k} \int_{\gamma(h,k)} \tag{15}$$

where $\gamma(h,k)$ is the upper arc of the Ford circle $\mathscr{C}(h,k)$. This allows us to work on each arc separately.

3. Changing variables:

The following change of variables maps the Ford circles $\mathscr{C}(h,k)$ onto the circle K of radius 1/2 with center $z = 1/2$:

$$z = -ik^2\left(\tau - \frac{h}{k}\right) \quad \leftrightarrow \quad \tau = \frac{h}{k} + \frac{iz}{k^2} \tag{16}$$

The points of tangency $\alpha_1(h,k)$ and $\alpha_2(h,k)$ of $\mathscr{C}(h,k)$ with the neighbouring circles $\mathscr{C}(h_1,k_1)$ and $\mathscr{C}(h_2,k_2)$ are mapped to the points $z_1(h,k)$ and $z_2(h,k)$ on K given by:

$$z_1(h,k) = \frac{k^2}{k^1+k_1^2} + i\frac{kk_1}{k^2+k_1^2} \qquad \text{and} \qquad z_2(h,k) = \frac{k^2}{k^1+k_2^2} - i\frac{kk_2}{k^2+k_2^2} \tag{17}$$

This is illustrated on Figure 2.

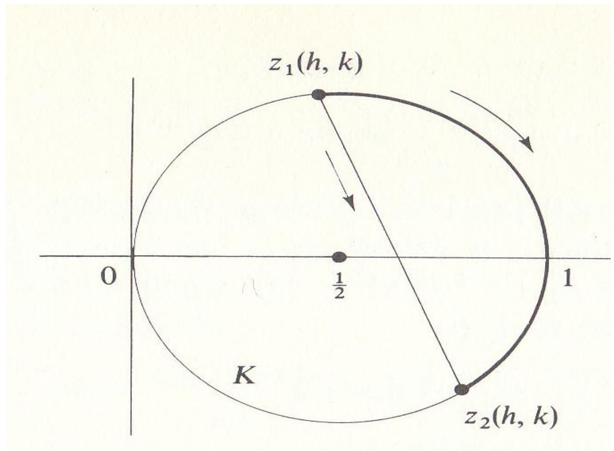

Figure 2:

The partition function is now written as:

$$\begin{aligned} p(n) &= \frac{1}{2\pi i}\sum_{h,k}\int_{\gamma(h,k)} \frac{F(e^{2\pi i\tau})}{e^{2\pi i\tau n}}d\tau && (18)\\ &= \sum_{h,k}\frac{i}{k^2}\exp\left(-\frac{2\pi inh}{k}\right)\int_{z_1(h,k)}^{z_2(h,k)}\exp\left(\frac{2n\pi z}{k^2}\right)F\left(\exp\left(\frac{2\pi ih}{k}-\frac{2\pi z}{k^2}\right)\right)dz && (19)\end{aligned}$$

Note that on K, $\mathrm{Re}(z) > 0$.

In order to perform the estimates in part 5 of the proof, the following facts are useful:

Theorem: If z is on the chord joining $z_1(h,k)$ and $z_2(h,k)$, we have $|z| < \frac{\sqrt{2}k}{N}$, and the length of that chord is $< 2\sqrt{2}k/N$.

Proof. We have:

$$|z_1|^2 = \frac{k^4+k^2k_1^2}{(k^2+k_1^2)^2} = \frac{k^2}{k^2+k_1^2} \quad \Longrightarrow \quad |z_1| = \frac{k}{\sqrt{k^2+k_1^2}} \tag{20}$$

Similarly $|z_2| = \frac{k}{\sqrt{k^2+k_2^2}}$. If z is on the chord joining $z_1(h,k)$ and $z_2(h,k)$, $|z| \le \max(|z_1|,|z_2|)$. Using:

$$\left(\frac{k+k_1}{2}\right)^2 \le \frac{k^2+k_1^2}{2} \quad \Longleftrightarrow \quad \frac{1}{\sqrt{k^2+k_1^2}} \le \frac{2}{k+k_1} \le \frac{\sqrt{2}}{N+1} \le \frac{\sqrt{2}}{N} \quad \Longleftrightarrow \quad |z_1| < \frac{\sqrt{2}k}{N} \tag{21}$$

And similarly $|z_2| < \frac{\sqrt{2}k}{N}$, giving $|z| < \frac{\sqrt{2}k}{N}$.
As for the length of the chord, it is $\le |z_1| + |z_2| < \frac{2\sqrt{2}k}{N}$.

□

4. DEDEKIND'S FUNCTIONAL EQUATION EXPRESSED IN TERMS OF F:

The Dedekind eta function is of central importance in many applications of elliptic modular functions to number theory. It was introduced by Dedekind in 1877 and is defined in the half-plane $H = \{\tau : \mathrm{Im}(\tau) > 0\}$ by the equation:

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}) \tag{22}$$

The functional equation satisfied by $\eta(\tau)$ is given in the following theorem, given without proof:

Theorem: If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, Γ being the modular group, and $c > 0$ we have

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \exp\left[\pi i\left(\frac{a+d}{12c}+s(-d,c)\right)\right]\sqrt{-i(c\tau+d)}\eta(\tau) \quad \text{where} \quad s(h,k) = \sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k}-\left[\frac{hr}{k}\right]-\frac{1}{2}\right)$$

In terms of F, this functional equation can be restated as:

Theorem: Let $F(t) = \prod_{k=1}^{\infty}\frac{1}{1-t^k}$ and

$$x = \exp\left(\frac{2\pi ih}{k}-\frac{2\pi z}{k^2}\right) \qquad x' = \exp\left(\frac{2\pi iH}{k}-\frac{2\pi}{z}\right) \tag{23}$$

where $\mathrm{Re}(z) > 0$, $k > 0$, $(h,k) = 1$ and $hH = -1 \bmod k$. Then

$$F(x) = \exp\left(\pi is(h,k)\right)\left(\frac{z}{k}\right)^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{\pi z}{12k^2}\right)F(x') \tag{24}$$

Note that when $|z|$ is small, we have

$$x = \exp\left(\frac{2\pi ih}{k}-\frac{2\pi z}{k^2}\right) \approx e^{2\pi ih/k} \qquad x' = \exp\left(\frac{2\pi iH}{k}-\frac{2\pi}{z}\right) \approx 0 \tag{25}$$

So that the behaviour of F near the singularity $e^{2\pi ih/k}$ is given by:

$$F(x) = \exp\left(\pi is(h,k)\right)\left(\frac{z}{k}\right)^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{\pi z}{12k^2}\right)\underbrace{F(x')}_{\approx 1} \tag{26}$$

Proof. First notice the following relation:

$$F(e^{2\pi i\tau}) = \prod_{k=1}^{\infty}\frac{1}{1-e^{2\pi ki\tau}} = \frac{e^{\pi i\tau/12}}{e^{\pi i\tau/12}\prod_{k=1}^{\infty}(1-e^{2\pi ki\tau})} = \frac{e^{\pi i\tau/12}}{\eta(\tau)} \tag{27}$$

Therefore:

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \exp\left[\pi i\left(\frac{a+d}{12c}+s(-d,c)\right)\right]\sqrt{-i(c\tau+d)}\eta(\tau) \tag{28}$$

$$\implies \frac{1}{\eta(\tau)} = \frac{1}{\eta\left(\frac{a\tau+b}{c\tau+d}\right)}\exp\left[\pi i\left(\frac{a+d}{12c}+s(-d,c)\right)\right]\sqrt{-i(c\tau+d)} \tag{29}$$

$$\implies F\left(\exp\left(2\pi i\tau\right)\right) = F\left(\exp\left(2\pi i\cdot\frac{a\tau+b}{c\tau+d}\right)\right)\exp\left(\frac{\pi}{12}i\left(\tau-\frac{a\tau+b}{c\tau+d}\right)\right) \tag{30}$$

$$\cdot\exp\left[\pi i\left(\frac{a+d}{12c}+s(-d,c)\right)\right]\sqrt{-i(c\tau+d)} \tag{31}$$

Choosing $a = H,\;\; c = k,\;\; d = -h,\;\; b = -\frac{hH+1}{k}$, and $\;\tau = \frac{iz+h}{k}$, then $\tau = \frac{\frac{i}{z}+H}{k}$ and

$$F\left(\exp\left(\frac{2\pi ih}{k}-\frac{2\pi z}{k}\right)\right) = F\left(\exp\left(\frac{2\pi iH}{k}-\frac{2\pi}{kz}\right)\right)z^{\frac{1}{2}}\exp\left(\frac{\pi}{12kz}-\frac{\pi z}{12k}+\pi is(h,k)\right) \tag{32}$$

Replacing z by z/k yields the result. □

We now have:

$$
\begin{aligned}
p(n) &= \sum_{h,k} \frac{i}{k^2} \exp\left(\frac{-2\pi inh}{k}\right) \int_{z_1(h,k)}^{z_2(h,k)} \exp\left(\frac{2n\pi z}{k^2}\right) F\left(\underbrace{\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right)}_{=x}\right) dz \\
&= \sum_{h,k} \frac{i}{k^2} \exp\left(\frac{-2\pi inh}{k}\right) \int_{z_1(h,k)}^{z_2(h,k)} \exp\left(\frac{2n\pi z}{k^2}\right) \exp\left(\pi is(h,k)\right) \left(\frac{z}{k}\right)^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F(x')dz \\
&= \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi is(h,k)\right) \int_{z_1(h,k)}^{z_2(h,k)} \exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F(x')dz
\end{aligned}
$$

Rewriting $F(x') = 1 + (F(x') - 1)$, we can split the integral into two parts:

$$
\begin{aligned}
p(n) &= \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi is(h,k)\right) \int_{z_1(h,k)}^{z_2(h,k)} \exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) \left(1 + (F(x') - 1)\right) dz \\
&= \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi is(h,k)\right) \underbrace{\int_{z_1(h,k)}^{z_2(h,k)} \exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) dz}_{=:\ I_1} \\
&\quad + \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi is(h,k)\right) \underbrace{\int_{z_1(h,k)}^{z_2(h,k)} \exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) \left(F(x') - 1\right) dz}_{=:\ I_2}
\end{aligned}
$$

5. Estimates for I_1 and I_2:

•We first estimate I_2, and start by moving the path of integration from the arc of K joining z_1 to z_2 to the chord joining z_1 and z_2. The integrand can then be bounded as follows:

$$
\begin{aligned}
& \left| \exp\left(\frac{2n\pi z}{k^2}\right) z^{1/2} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) \left[F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right) - 1\right] \right| && (33) \\
=\ & \left| \exp\left(\frac{2n\pi z}{k^2}\right) z^{1/2} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) \left[\sum_{m=0}^{\infty} p(m) \exp\left(\frac{2\pi iHm}{k}\right) \exp\left(-\frac{2\pi m}{z}\right) - 1\right] \right| && (34) \\
\leq\ & \exp\left(\frac{2n\pi \overbrace{\mathrm{Re}(z)}^{\leq 1}}{k^2}\right) |z|^{1/2} \exp\left(\frac{\pi}{12}\mathrm{Re}(\frac{1}{z}) - \overbrace{\frac{\pi \mathrm{Re}(z)}{12k^2}}^{\text{drop}}\right) \sum_{m=1}^{\infty} p(m) \underbrace{\left|\exp\left(\frac{2\pi iHm}{k}\right)\right|}_{=1} \underbrace{\left|\exp\left(-\frac{2\pi m}{z}\right)\right|}_{\leq \exp(-2\pi m \mathrm{Re}(1/z))} && (35) \\
\leq\ & \exp\left(\frac{2n\pi}{k^2}\right) |z|^{1/2} \exp\left(\frac{\pi}{12}\mathrm{Re}(\frac{1}{z})\right) \sum_{m=1}^{\infty} p(m) \exp\left(-2\pi m \mathrm{Re}(\frac{1}{z})\right) && (36) \\
<\ & \exp\left(2n\pi\right) |z|^{1/2} \sum_{m=1}^{\infty} \overbrace{p(m)}^{<p(24m-1)} \exp\left(-2\pi\left(m - 1/24\right) \overbrace{\mathrm{Re}(\frac{1}{z})}^{\geq 1}\right) && (37) \\
<\ & \exp\left(2n\pi\right) |z|^{1/2} \sum_{m=1}^{\infty} p(24m-1) \exp\left(-2\pi\left(m - 1/24\right)\right) && (38) \\
:=\ & c|z|^{1/2} && (39)
\end{aligned}
$$

where c is a constant that does not depend on z nor N. Since $|z| < \sqrt{2}k/N$ on the chord, and the length of the chord is $< 2\sqrt{2}k/N$, we have:

$$|I_2| \;\leq\; \int_{z_1(h,k)}^{z_2(h,k)} \left|\exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)(F(x')-1)\right| dz \;<\; \frac{2\sqrt{2}k}{N} \cdot c \left(\frac{\sqrt{2}k}{N}\right)^{\frac{1}{2}} \;:=\; C\left(\frac{k}{N}\right)^{\frac{3}{2}} \quad (40)$$

$$\implies \left|\sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi i s(h,k)\right) I_2\right| \;<\; \sum_{k=1}^{N} \sum_{0\leq h<k} Ck^{-\frac{5}{2}} \left(\frac{k}{N}\right)^{\frac{3}{2}} = \mathscr{O}(N^{-\frac{1}{2}}) \quad (41)$$

•We now estimate I_1: here, instead of only integrating over the arc of K joining z_1 to z_2, we integrate over the whole circle K. Going *clockwise* around the circle:

$$\oint_K = \int_0^{z_2(h,k)} + \int_{z_2(h,k)}^{z_1(h,k)} + \int_{z_1(h,k)}^{0} =: J_2 + \int_{z_2(h,k)}^{z_1(h,k)} + J_1 \qquad \implies \qquad \int_{z_2(h,k)}^{z_1(h,k)} = \oint_K -J_1 - J_2 \quad (42)$$

- Estimating J_1:

$$\left|\exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)\right| \;=\; \exp\left(\frac{2n\pi \overbrace{\mathrm{Re}(z)}^{\leq 1}}{k^2}\right) \left(\underbrace{|z|}_{\leq \frac{\sqrt{2}k}{N}}\right)^{\frac{1}{2}} \exp\left(\frac{\pi}{12} \underbrace{\mathrm{Re}\left(\frac{1}{z}\right)}_{=1} \underbrace{-\frac{\pi}{12k^2}\mathrm{Re}(z)}_{\text{drop}}\right)$$

$$\leq\; \frac{\exp(2n\pi) \cdot 2^{\frac{1}{4}} \cdot k^{\frac{1}{2}} \cdot \exp\left(\frac{\pi}{12}\right)}{N^{\frac{1}{2}}} \;:=\; C_1 \left(\frac{k}{N}\right)^{\frac{1}{2}}$$

$$\implies |J_1| \;\leq\; \int_0^{z_1(h,k)} \left|\exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)\right| dz \;\leq\; \pi z_1(h,k) \cdot C_1 \left(\frac{k}{N}\right)^{\frac{1}{2}} \quad (43)$$

$$<\; \pi\sqrt{2}\frac{k}{N} \cdot C_1 \left(\frac{k}{N}\right)^{\frac{1}{2}} \;=:\; \tilde{C}_1 \left(\frac{k}{N}\right)^{\frac{3}{2}} \quad (44)$$

- Similarly: $|J_2| \leq \tilde{C}_2 \left(\frac{k}{N}\right)^{\frac{3}{2}}$.

Therefore

$$\begin{aligned}
&\sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi i s(h,k)\right) I_1 \\
=\;& \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi i s(h,k)\right) \int_{z_1(h,k)}^{z_2(h,k)} \exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) dz \\
=\;& \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi i s(h,k)\right) \left(\oint_K \exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) dz - J_1 - J_2\right) \\
=\;& \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi i s(h,k)\right) \oint_K \exp\left(\frac{2n\pi z}{k^2}\right) z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) dz \\
&- \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi i s(h,k)\right) \cdot J_1 - \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi i s(h,k)\right) \cdot J_2
\end{aligned}$$

and by the above:

$$\begin{aligned}
&\left|\sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi i s(h,k)\right) \cdot J_1 + \sum_{h,k} ik^{-\frac{5}{2}} \exp\left(\frac{-2\pi inh}{k}\right) \exp\left(\pi i s(h,k)\right) \cdot J_2\right| \\
\leq\;& \sum_{k=1}^{N} \sum_{0\leq h<k} k^{-\frac{5}{2}} \cdot |J_1| + \sum_{k=1}^{N} \sum_{0\leq h<k} k^{-\frac{5}{2}} \cdot |J_2| \\
<\;& \sum_{k=1}^{N} \sum_{0\leq h<k} k^{-\frac{5}{2}} \cdot \tilde{C}_1 \left(\frac{k}{N}\right)^{\frac{3}{2}} + \sum_{k=1}^{N} \sum_{0\leq h<k} k^{-\frac{5}{2}} \cdot \tilde{C}_2 \left(\frac{k}{N}\right)^{\frac{3}{2}} = \mathscr{O}(N^{-\frac{1}{2}})
\end{aligned}$$

Thus:

$$
\begin{aligned}
p(n) &= \sum_{h,k} ik^{-\frac{5}{2}}\exp\left(\frac{-2\pi inh}{k}\right)\exp\left(\pi is(h,k)\right)I_1 + \sum_{h,k} ik^{-\frac{5}{2}}\exp\left(\frac{-2\pi inh}{k}\right)\exp\left(\pi is(h,k)\right)I_2 \\
&= \sum_{k=1}^{N}\sum_{0\le h<k} ik^{-\frac{5}{2}}\exp\left(\frac{-2\pi inh}{k}\right)\exp\left(\pi is(h,k)\right)\oint_K \exp\left(\frac{2n\pi z}{k^2}\right)z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{\pi z}{12k^2}\right)dz + \mathscr{O}(N^{-\frac{1}{2}}) \\
&= \sum_{k=1}^{N}\sum_{0\le h<k} ik^{-\frac{5}{2}}\exp\left(\frac{-2\pi inh}{k}\right)\exp\left(\pi is(h,k)\right)\oint_K z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{2\pi z}{k^2}\left(n-\frac{1}{24}\right)\right)dz + \mathscr{O}(N^{-\frac{1}{2}})
\end{aligned}
$$

6. Getting the final answer using Bessel functions:

We start by letting N, the number of circles used to construct Rademacher's path, go to $+\infty$. Defining $A_k(n) = \sum_{0\le h<k}\exp\left(\pi is(h,k) - 2\pi in\frac{h}{k}\right)$:

$$
\begin{aligned}
p(n) &= \lim_{N\to+\infty}\left[\sum_{k=1}^{N}\sum_{0\le h<k} ik^{-\frac{5}{2}}\exp\left(\frac{-2\pi inh}{k}\right)\exp\left(\pi is(h,k)\right)\oint_K z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{2\pi z}{k^2}\left(n-\frac{1}{24}\right)\right)dz + \mathscr{O}(N^{-\frac{1}{2}})\right] \\
&= i\sum_{k=1}^{+\infty} k^{-\frac{5}{2}}A_k(n)\oint_K z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{2\pi z}{k^2}\left(n-\frac{1}{24}\right)\right)dz
\end{aligned}
$$

We then perform two successive changes of variables: First let $w = \frac{1}{z}$ so that $\mathrm{d}z = -\frac{1}{w^2}\mathrm{d}w$. This maps the circle to the infinite line $\mathrm{Re}(z) = 1$:

$$
\begin{aligned}
p(n) &= i\sum_{k=1}^{+\infty} k^{-\frac{5}{2}}A_k(n)\oint_K z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{2\pi z}{k^2}\left(n-\frac{1}{24}\right)\right)dz && (45)\\
&= i\sum_{k=1}^{+\infty} k^{-\frac{5}{2}}A_k(n)\int_{1-\infty i}^{1+\infty i}\left(\frac{1}{w}\right)^{\frac{1}{2}}\exp\left(\frac{\pi w}{12}-\frac{2\pi}{k^2}\left(n-\frac{1}{24}\right)\frac{1}{w}\right)\left(-\frac{1}{w^2}\right)dw && (46)\\
&= -i\sum_{k=1}^{+\infty} k^{-\frac{5}{2}}A_k(n)\int_{1-\infty i}^{1+\infty i} w^{-\frac{5}{2}}\exp\left(\frac{\pi w}{12}-\frac{2\pi}{k^2}\left(n-\frac{1}{24}\right)\frac{1}{w}\right)dw && (47)
\end{aligned}
$$

Secondly, let $t = \frac{\pi w}{12}$ or $w = \frac{12t}{\pi}$ to get:

$$
\begin{aligned}
p(n) &= -i\sum_{k=1}^{+\infty} k^{-\frac{5}{2}}A_k(n)\int_{1-\infty i}^{1+\infty i} w^{-\frac{5}{2}}\exp\left(\frac{\pi w}{12}-\frac{2\pi}{k^2}\left(n-\frac{1}{24}\right)\frac{1}{w}\right)dw && (48)\\
&= -i\sum_{k=1}^{+\infty} k^{-\frac{5}{2}}A_k(n)\int_{\frac{\pi}{12}-\infty i}^{\frac{\pi}{12}+\infty i}\left(\frac{12t}{\pi}\right)^{-\frac{5}{2}}\exp\left(t-\frac{2\pi}{k^2}\left(n-\frac{1}{24}\right)\frac{\pi}{12t}\right)\left(\frac{12}{\pi}\right)dt && (49)\\
&= -i\left(\frac{\pi}{12}\right)^{\frac{3}{2}}\sum_{k=1}^{+\infty} k^{-\frac{5}{2}}A_k(n)\int_{\frac{\pi}{12}-\infty i}^{\frac{\pi}{12}+\infty i} t^{-\frac{5}{2}}\exp\left(t-\frac{\pi^2}{6k^2}\left(n-\frac{1}{24}\right)\frac{1}{t}\right)dt && (50)\\
&= 2\pi\left(\frac{\pi}{12}\right)^{\frac{3}{2}}\sum_{k=1}^{+\infty} k^{-\frac{5}{2}}A_k(n)\frac{1}{2\pi i}\int_{\frac{\pi}{12}-\infty i}^{\frac{\pi}{12}+\infty i} t^{-\frac{5}{2}}\exp\left(t-\frac{\pi^2}{6k^2}\left(n-\frac{1}{24}\right)\frac{1}{t}\right)dt && (51)
\end{aligned}
$$

Noting that if $c > 0$ and $\mathrm{Re}(\nu) > 0$:

$$
I_\nu(z) = \frac{(z/2)^\nu}{2\pi i}\int_{c-\infty i}^{c+\infty i} t^{-\nu-1}e^{t+(z^2/4t)}dt \qquad (52)
$$

where $I_\nu(z) = i^{-\nu} J_\nu(iz)$, we have that if we let $\frac{z}{2} = \left(\frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\right)^{\frac{1}{2}}$, then:

$$
\begin{aligned}
p(n) = 2\pi\left(\frac{\pi}{12}\right)^{\frac{3}{2}} \sum_{k=1}^{+\infty} k^{-\frac{5}{2}} A_k(n) \frac{1}{2\pi i} \int_{\frac{\pi}{12}-\infty i}^{\frac{\pi}{12}+\infty i} t^{-\frac{5}{2}} \exp\left(t - \frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\frac{1}{t}\right) \mathrm{d}t \\
= \ & 2\pi\left(\frac{\pi}{12}\right)^{\frac{3}{2}} \ \sum_{k=1}^{+\infty} k^{-\frac{5}{2}} A_k(n) \cdot \frac{1}{\left(\frac{\pi^2}{6k^2}\left(n-\frac{1}{24}\right)\right)^{\frac{3}{4}}} \cdot \frac{\left(\frac{\pi^2}{6k^2}\left(n-\frac{1}{24}\right)\right)^{\frac{3}{4}}}{2\pi i} \int_{\frac{\pi}{12}-\infty i}^{\frac{\pi}{12}+\infty i} t^{-\frac{3}{2}-1} \exp\left(t - \frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\frac{1}{t}\right) \mathrm{d}t \\
= \ & 2\pi\left(\frac{\pi}{12}\right)^{\frac{3}{2}} \ \sum_{k=1}^{+\infty} k^{-\frac{5}{2}} A_k(n) \cdot \frac{6^{\frac{3}{4}} k^{\frac{3}{2}}}{\pi^{\frac{3}{2}}\left(n-\frac{1}{24}\right)^{\frac{3}{4}}} \cdot \underbrace{\frac{\left(\frac{z}{2}\right)^{\frac{3}{2}}}{2\pi i} \int_{\frac{\pi}{12}-\infty i}^{\frac{\pi}{12}+\infty i} t^{-\frac{3}{2}-1} \exp\left(t - \frac{z^2}{4t}\right) \mathrm{d}t}_{=I_{3/2}(z)} \\
= \ & \frac{2\pi}{\left(24\left(n-\frac{1}{24}\right)\right)^{\frac{3}{4}}} \ \sum_{k=1}^{+\infty} A_k(n) k^{-1} I_{3/2}(z)
\end{aligned}
$$

But Bessel functions of half-order can be reduced to elementary functions:

$$
I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz}\left(\frac{\sinh z}{z}\right) \tag{53}
$$

We rewrite everything in terms of n:

$$
\frac{z}{2} = \left(\frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\right)^{\frac{1}{2}} \iff z = \frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)} \iff n = \frac{1}{24} + \frac{3}{2}\left(\frac{zk}{\pi}\right)^{\frac{1}{2}} \tag{54}
$$

$$
\Longrightarrow \frac{d}{dz} \ = \ \frac{dn}{dz} \cdot \frac{d}{dn} \ = \ 3z\frac{k^2}{\pi^2} \cdot \frac{d}{dn} \ = \ 3\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\frac{k^2}{\pi^2}\frac{d}{dn} \tag{55}
$$

Also:

$$
\begin{aligned}
I_{3/2}(z) \ &= \ \sqrt{\frac{2z}{\pi}} \ \frac{d}{dz} \ \left(\frac{\sinh z}{z}\right) \\
\Longrightarrow \quad I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right) \ &= \ \sqrt{\frac{2}{\pi} \cdot \frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}} \cdot 3\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\frac{k^2}{\pi^2}\frac{d}{dn} \ \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}}\right) \\
&= \ \frac{2^{\frac{1}{2}}}{k^{\frac{1}{2}}}\frac{2^{\frac{1}{4}}}{3^{\frac{1}{4}}}\left(n - \frac{1}{24}\right)^{\frac{1}{4}} \cdot 3\frac{\pi}{k}\frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(n - \frac{1}{24}\right)^{\frac{1}{2}} \frac{k^2}{\pi^2} \cdot \frac{k}{\pi}\frac{3^{\frac{1}{2}}}{2^{\frac{1}{2}}}\frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right) \\
&= \ \frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{2}} \cdot 3 \cdot 3^{\frac{1}{2}}}{3^{\frac{1}{4}} \cdot 3^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}} \cdot \frac{\pi}{\pi^2 \cdot \pi} \cdot \frac{k^2 \cdot k}{k^{\frac{1}{2}} \cdot k} \cdot \left(n - \frac{1}{24}\right)^{\frac{3}{4}} \ \frac{d}{dn} \ \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right) \\
&= \ 2^{\frac{3}{4}} \cdot 3^{\frac{3}{4}} \cdot \frac{1}{\pi^2} \cdot k^{\frac{3}{2}} \cdot \left(n - \frac{1}{24}\right)^{\frac{3}{4}} \ \frac{d}{dn} \ \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right)
\end{aligned}
$$

So that:

$$
p(n) \ = \ \frac{2\pi}{\left(24\left(n-\frac{1}{24}\right)\right)^{\frac{3}{4}}} \ \sum_{k=1}^{+\infty} A_k(n) k^{-1} I_{3/2}(z)
$$

$$
\begin{aligned}
&= \frac{2\pi}{2^{\frac{9}{4}}3^{\frac{3}{4}}\left(n-\frac{1}{24}\right)^{\frac{3}{4}}} \sum_{k=1}^{+\infty} A_k(n)\frac{1}{k}\cdot 2^{\frac{3}{4}}\cdot 3^{\frac{3}{4}}\cdot \frac{1}{\pi^2}\cdot k^{\frac{3}{2}}\cdot \left(n-\frac{1}{24}\right)^{\frac{3}{4}} \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right) \\
&= \frac{2\cdot 2^{\frac{3}{4}}\cdot 3^{\frac{3}{4}}}{2^{\frac{9}{4}}\cdot 3^{\frac{3}{4}}}\cdot\frac{\pi}{\pi^2}\cdot\frac{(n-\frac{1}{24})^{\frac{3}{4}}}{(n-\frac{1}{24})^{\frac{3}{4}}} \sum_{k=1}^{+\infty} A_k(n)\cdot\frac{k^{\frac{3}{2}}}{k}\cdot\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right) \\
&= \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right)
\end{aligned}
$$

as required.

□

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