MATH 566 LECTURE NOTES 5: THE RIEMANN MAPPING THEOREM

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1. Introduction

We start with a definition.

Definition 1. A mapping $\phi : \Omega \to \Sigma$ is called *biholomorphic* onto Σ, or *conformal*, if $\phi \in \mathcal{O}(\Omega)$ and ϕ is invertible with $\phi^{-1} \in \mathcal{O}(\Sigma)$. If there is a conformal mapping between Ω and Σ, then they are said to be *conformally equivalent* domains.

Biholomorphic mappings are important because they can be used to "plant" complex analysis of one domain onto another domain: in the setting of the preceding definition, if $f \in \mathcal{O}(\Sigma)$ then $f \circ \phi \in \mathcal{O}(\Omega)$, and if $g \in \mathcal{O}(\Omega)$ then $g \circ \phi^{-1} \in \mathcal{O}(\Sigma)$. Conformal equivalence is an equivalence relation on the space of all domains (or more generally, Riemann surfaces), and so in principle it suffices to study one representative from each of those equivalence classes (called *conformal classes*). Therefore, identifying conformal classes is a fundamental problem in complex analysis. The Riemann mapping theorem characterizes the conformal class of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which turns out to be the collection of all simply connected domains (except $\mathbb{C}$ itself). In the following, we discuss the Riemann mapping theorem and some other topics related to it.

2. Mapping properties of holomorphic functions

Let f be a nonconstant holomorphic function on a neighbourhood of $0 \in \mathbb{C}$, with $f(0) = 0$. Obviously we can write $f(z) = z^n g(z)$ with $g(0) \neq 0$, and $n \geq 1$. Moreover, there is $\delta > 0$ so small that the *both* functions f and f' are nowhere vanishing in the punctured closed disk $\overline{D_\delta} \setminus \{0\}$. Let $w \in \mathbb{C}^\times$ be such that $|w| < \min\limits_{|z|=\delta} |f(z)|$. Then, by Rouché's theorem, the function $z \mapsto f(z) - w$ has exactly n zeroes in D_δ. Furthermore, those zeroes must be in $D_\delta^\times$ because $w \neq 0$, and since f' does not vanish in $D_\delta^\times$, the zeroes must be simple. We conclude that each point in the disk D_R of radius $R = \min\limits_{|z|=\delta} |f(z)|$ has exactly n distinct points in D_δ as its preimage set under f. Since f is continuous, $f^{-1}(D_R) \subseteq D_\delta$ is an open neighbourhood of 0, so f is n-to-1 on a neighbourhood of 0. In particular, recalling that $n = 1$ is equivalent to $f'(0) \neq 0$, local injectivity implies nonvanishing of the first derivative and vice versa. This substantially clarifies local mapping properties of holomorphic functions.

Theorem 2. *Let f be a holomorphic function on a neighbourhood of $c \in \mathbb{C}$.*

(a) If f is injective on a neighbourhood of c, then $f'(c) \neq 0$.
(b) Conversely, if $f'(c) \neq 0$, then f is injective on a neighbourhood of c.
(c) More generally, if f is not constant, then on a neighbourhood of c, it can be written as

$$f(z) = f(c) + \varphi(z)^n,$$

where φ is a holomorphic injection and $n = \mathrm{Ord}(f', c) + 1$.

Date: December 3, 2010.

Proof. We may assume, without loss of generality, that $c = 0$ and $f(0) = 0$. Claims (a) and (b) have been proven above. In the notations of the paragraph preceding this theorem, since g is nonvanishing on a neighbourhood of 0, there is a holomorphic ψ such that $\psi(z)^n = g(z)$ on a neighbourhood of 0. Then with $\varphi(z) = z\psi(z)$, we have $\varphi'(0) = \psi(0) \neq 0$. □

Theorem 3 (Inverse function theorem). *With some disk D, let $f \in \mathcal{O}(\overline{D})$ be injective, and let $\Sigma = f(D)$. Then the inverse function $f^{-1} : \Sigma \to D$ is holomorphic, and we have*

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{\partial D}\frac{zf'(z)\mathrm{d}z}{f(z)-w}, \qquad w \in \Sigma.$$

In particular, if $f \in \mathcal{O}(\Omega)$ is injective on a domain Ω, then its inverse f^{-1} is holomorphic.

Proof. Let $\zeta \in D$, and let $w = f(\zeta)$. Since f is injective, $z \mapsto f(z) - w$ has one simple zero at $z = \zeta$ in D. We apply the generalized argument principle with $g(z) = z$, to obtain

$$\zeta = \frac{1}{2\pi i}\int_{\partial D}\frac{zf'(z)\mathrm{d}z}{f(z)-w},$$

establishing the formula. By holomoprhy of integral then we get the holomorphy of f^{-1}.

For the second part of the theorem, for each $w \in f(\Omega)$, we apply the first part of the theorem to a disk centred at $f^{-1}(w)$ whose closure lies in Ω, to conclude that f^{-1} is holomorphic on a neighbourhood of w. □

Holomorphic mappings generally preserve angles and orientations, in the sense we discuss below. By definition, a curve $\gamma : [-1, 1] \to \mathbb{C}$ is differentiable at the parameter value 0 if there exists $\tau \in \mathbb{C}$ such that

$$\gamma(\varepsilon) = \gamma(0) + \varepsilon\tau + o(|\varepsilon|),$$

and in this situation τ is called the *tangent* of γ at $\gamma(0)$. Now if ϕ is a holomorphic function on a neighbourhood of $\gamma(0)$, then there is $\lambda \in \mathbb{C}$ such that

$$\phi(z + h) = \phi(z) + \lambda h + o(|h|),$$

and moreover $\phi \circ \gamma$ is a curve passing through and differentiable at $\phi(\gamma(0))$. By combining the preceding two definitions we can write

$$\phi(\gamma(\varepsilon)) = \phi(\gamma(0)) + \lambda\tau\varepsilon + o(\varepsilon),$$

revealing that the tangent of $\phi \circ \gamma$ at $\phi(\gamma(0))$ is $\lambda\tau$. In particular, if $\lambda \equiv \phi'(\gamma(0)) \neq 0$, or equivalently if ϕ is injective on a neighbourhood of $\gamma(0)$, then under ϕ all tangents are rotated by the same angle, so the angle between any two curves intersecting at $\gamma(0)$, together with its orientation, is preserved under ϕ. This is almost the general picture, since the derivative ϕ' is holomorphic and therefore its zeroes form a discrete set, provided that ϕ is not constant. What happens at the zeroes of ϕ' is also simple to understand. We know from Theorem 2(c) that at the zeroes of ϕ' of order m, ϕ is equal to a holomorphic injection followed by the polynomial map $z \mapsto z^{m+1}$, hence the angles are simply multiplied by $m + 1$. Let us summarize these observations into the following informal remark, where we call the zeroes of ϕ' the *critical points* of ϕ.

Remark 4. Near any of its non-critical points, a holomorphic mapping is one-to-one, and angle- and orientation preserving. Near a critical point, the mapping is n-to-1 for some integer n (that may vary from critical point to critical point), and at the critical point itself, the angles are multiplied by n. The critical points form a discrete set.

3. Automorphism groups

Definition 5. Given a connected open set Ω, its *automorphism group*, denoted by $\operatorname{Aut}\Omega$, is the set of biholomorphic self-maps of Ω.

One can easily see that *affine maps*, that are the maps of the form $z \mapsto az + b$ with $a \in \mathbb{C}^\times$ and $b \in \mathbb{C}$, are in the automorphism group of the complex plane $\mathbb{C}$. In fact, those are the only maps in $\operatorname{Aut}\mathbb{C}$.

Lemma 6. $\operatorname{Aut}\mathbb{C} = \{z \mapsto az + b : a \in \mathbb{C}^\times,\ b \in \mathbb{C}\}$.

Proof. Let $\phi \in \operatorname{Aut}\mathbb{C}$. Then $\phi(\mathbb{D})$ is an open set, and by injectivity, near ∞, ϕ does not take any values from $\phi(\mathbb{D})$. So it follows from the Casorati-Weierstrass theorem that $\infty \in \hat{\mathbb{C}}$ is not an essential singularity, which means that ϕ is a polynomial. But since ϕ' has no zero in $\mathbb{C}$, we have $\phi' = \text{const} \neq 0$, which concludes the proof. ☐

Let us turn to the automorphisms of $\hat{\mathbb{C}}$. Given a 2×2 complex matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the function $F_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by $F_A(z) = \frac{az+b}{cz+d}$. Its derivative is given by $F_A'(z) = \frac{ad-bc}{(cz+d)^2}$, so a necessary condition for F_A to be locally injective is $ad - bc \equiv \det A \neq 0$. Such functions F_A with $\det A \neq 0$ are called *fractional linear transformations*. If $c = 0$ (and so $ad \neq 0$) these reduce to affine maps, which now can be thought of as fractional linear transformations that leave $\infty \in \hat{\mathbb{C}}$ fixed. In general, it holds that $F_A(\infty) = a/c$ and $F_A(-d/c) = \infty$. Moreover, we have $F_A(1/w) = F_{A'}(w)$ where A' is the matrix obtained from A by interchanging its columns, and $1/F_A(z) = F_{A''}(z)$ where A'' is the matrix obtained from A by interchanging its rows. Since $\det A' = \det A'' = \det A$, it follows that F_A is locally injective everywhere in $\hat{\mathbb{C}}$. One can easily check that the inverse of F_A is equal to $F_{A^{-1}}$, thus implying that $F_A \in \operatorname{Aut}\hat{\mathbb{C}}$. In fact, recalling that $\mathrm{GL}(2,\mathbb{C}) = \{A \in \mathbb{C}^{2\times 2} : \det A \neq 0\}$, the mapping $A \mapsto F_A$ is a group homomorphism between $\mathrm{GL}(2,\mathbb{C})$ and $\operatorname{Aut}\hat{\mathbb{C}}$ with the kernel given by nonzero complex multiples of the identity matrix. In other words, we have $F_{AB} = F_A \circ F_B$ and $F_{sA} = F_A$ for $s \in \mathbb{C}^\times$. The following lemma says that this homomorphism is surjective, i.e., the automorphisms of $\hat{\mathbb{C}}$ are exactly the fractional linear transformations.

Lemma 7. $\operatorname{Aut}\hat{\mathbb{C}} = \{F_A : A \in \mathrm{GL}(2,\mathbb{C})\}$.

Proof. Let $\phi \in \operatorname{Aut}\hat{\mathbb{C}}$. So ϕ must be a rational function, which can be written in the form

$$\phi(z) = \frac{a \cdot (z - \alpha_1) \cdots (z - \alpha_n)}{(z - \beta_1) \cdots (z - \beta_m)},$$

with $\alpha_j \neq \beta_k$ for all j, k. Then we have $\phi(\alpha_j) = 0$ for all j, and $\phi(\beta_k) = \infty$ for all k. Since ϕ is invertible, this forces $n = 1$ and $m = 1$. ☐

One useful property of fractional linear transformations is that one can freely specify the images of any three distinct points. Namely, if α, β, γ are three distinct points of $\hat{\mathbb{C}}$, then

$$\phi(z) = \frac{(z-\alpha)(\beta-\gamma)}{(z-\gamma)(\beta-\alpha)},$$

maps α to 0, β to 1, and γ to ∞. Now by composing this with the inverse of another such mapping, the triple (α, β, γ) can be mapped to any triple $(\alpha', \beta', \gamma')$ as long as α', β', γ' are distinct.

4. Schwarz's lemma

Lemma 8 (Schwarz). *Let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic with $\phi(0) = 0$. Then we have $|\phi(z)| \leq |z|$ for $z \in \mathbb{D}$ and $\phi'(0) \leq 1$. Moreover, if either $|\phi(z)| = |z|$ for some $z \neq 0$ or $|\phi'(0)| = 1$, then $\phi(z) = \lambda z$ for some $\lambda \in \partial\mathbb{D}$.*

Proof. This is a direct application of the maximum principle. Let

$$g(z) = \begin{cases} \phi(z)/z, & \text{if } z \neq 0, \\ \phi'(0), & \text{if } z = 0. \end{cases}$$

Then we have $g \in \mathcal{O}(\mathbb{D})$, and $|g| \leq \frac{1}{1-\varepsilon}$ on $\partial D_{1-\varepsilon}$ for $\varepsilon > 0$. The maximum principle gives $|g| \leq \frac{1}{1-\varepsilon}$ in $D_{1-\varepsilon}$, and letting $\varepsilon \to 0$, we infer $|g| \leq 1$ in $\mathbb{D}$. This is the first claim of the lemma. In terms of g, the hypothesis of the second claim is simply that $|g(z)| = 1$ for some $z \in \mathbb{D}$. But this implies by the maximum principle that $g = \text{const} = \lambda$ with $|\lambda| = 1$, which establishes the second claim. □

As a nontrivial application of Schwarz's lemma, below we characterize the automorphisms of the unit disk. Our initial candidates will be a special class of fractional linear transformations, namely, maps of the form

$$B_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}, \qquad \text{with} \quad |\alpha| < 1.$$

These maps are called the *Blaschke factors*. Since $|1 - z\bar{\alpha}| > 0$ for $z \in \mathbb{D}$, we have $B_\alpha \in \mathcal{O}(\mathbb{D})$. Moreover, from that

$$|B_\alpha(z)| = \left| \frac{z - \alpha}{1 - \bar{\alpha} z} \right| = \left| \bar{z} \frac{z - \alpha}{1 - \bar{\alpha} z} \right| = \left| \frac{1 - \alpha \bar{z}}{1 - \bar{\alpha} z} \right| = 1, \qquad \text{for} \quad z \in \partial\mathbb{D},$$

and that B_α is not constant, we infer by the maximum principle that $|B_\alpha(z)| < 1$ for $z \in \mathbb{D}$. Clearly B_α has the inverse $B_\alpha^{-1} = B_{-\alpha}$ on $\mathbb{D}$, hence $B_\alpha \in \operatorname{Aut}\mathbb{D}$. Let us also make a note of the simple properties $B_\alpha(\alpha) = 0$ and $B_\alpha(0) = -\alpha$.

Lemma 9. $\operatorname{Aut}\mathbb{D} = \{z \mapsto \lambda B_\alpha(z) : \alpha \in \mathbb{D}, \lambda \in \partial\mathbb{D}\}$.

Proof. First we prove that an automorphism of $\mathbb{D}$ that leaves 0 fixed must be a rigid rotation around 0. Let $\varphi \in \operatorname{Aut}\mathbb{D}$ with $\varphi(0) = 0$. Then Schwarz's lemma applied to φ and to φ^{-1} implies that $|\varphi(z)| \leq |z|$ and $|\varphi^{-1}(z)| \leq |z|$ for $z \in \mathbb{D}$. The latter can be written as $|z| = |\varphi^{-1}(\varphi(z))| \leq |\varphi(z)|$ for $z \in \mathbb{D}$, which, combined with the former, implies that $|\varphi(z)| = |z|$ for $z \in \mathbb{D}$. Now yet another application of Schwarz's lemma guarantees that $\varphi(z) = \lambda z$ for some $\lambda \in \partial\mathbb{D}$.

For general $\psi \in \operatorname{Aut}\mathbb{D}$, we consider the map $\varphi = \psi \circ B_{-\alpha}$ with $\alpha = \psi^{-1}(0)$. Then we have $\varphi \in \operatorname{Aut}\mathbb{D}$ and $\varphi(0) = \psi(B_{-\alpha}(0)) = \psi(\alpha) = 0$, and so by the preceding paragraph φ is a rotation around 0. The proof is complete upon recalling $\psi = \varphi \circ B_{-\alpha}^{-1} = \varphi \circ B_\alpha$. □

5. The Riemann mapping theorem

When is an open set $\Omega \subset \mathbb{C}$ conformally equivalent to the unit disk $\mathbb{D}$? One can exclude the case $\Omega = \mathbb{C}$ right away because by Liouville's theorem, any holomorphic function $\phi : \mathbb{C} \to \mathbb{D}$ must be constant. Also, since biholomorphic mappings are in particular homeomorphisms, Ω must be homeomorphic to $\mathbb{D}$, so a necessary condition is that Ω must be simply connected. The all-important Riemann mapping theorem says that these conditions are also sufficient. Namely, any simply connected, open proper subset of $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$. Before stating the Riemann mapping theorem, we introduce a definition.

Definition 10. A connected open set $\Omega \subset \mathbb{C}$ is said to have the *square-root property*, if for any zero-free function $f \in \mathcal{O}(\Omega)$, there exists $g \in \mathcal{O}(\Omega)$ such that $g(z)^2 = f(z)$ for $z \in \Omega$.

We have seen that if Ω is simply connected, then it has the above-defined property.

Theorem 11. *Let $\Omega \subsetneq \mathbb{C}$ have the square-root property, and let $\alpha \in \Omega$. Then there is a unique biholomorphic mapping $\phi : \Omega \to \mathbb{D}$ such that $\phi(\alpha) = 0$ and $\phi'(\alpha) > 0$.*

Proof. We explain here the main structure of the proof, deferring certain specific results to the three subsections below. Let us define the set

$$\Phi = \{\varphi : \Omega \to \mathbb{D} \text{ injective, holomorphic, and } \varphi(\alpha) = 0\}.$$

That this set is nonempty is proven in §5.1. Then we fix a point $\beta \in \Omega$ with $\beta \neq \alpha$, and look for maximizers of $|\varphi(\beta)|$ among $\varphi \in \Phi$. Let $B = \sup\{|\varphi(\beta)| : \varphi \in \Phi\}$, and let $\{\phi_n\} \subset \Phi$ be a sequence such that $|\phi_n(\beta)| \to B$ as $n \to \infty$. Since the functions ϕ_n map Ω into $\mathbb{D}$, we have $\|\phi_n\|_\Omega \leq 1$ for all n. By Montel's theorem that is proven in §5.2 below, $\{\phi_n\}$ has a subsequence that converges locally uniformly on Ω. Let us denote this limit function by $\phi : \Omega \to \mathbb{C}$, which by construction satisfies $\phi(\beta) = B$. The Weierstrass convergence theorem guarantees that ϕ is holomorphic, and the Hurwitz injection theorem implies that ϕ is injective, and moreover that the range is in $\mathbb{D}$, i.e., that $\phi(\Omega) \subseteq \mathbb{D}$. Furthermore, by the inverse function theorem, $\phi^{-1} : \phi(\Omega) \to \Omega$ is holomorphic. We prove that $\phi(\Omega) = \mathbb{D}$ in §5.3, so that the existence of a biholomorphism $\phi : \Omega \to \mathbb{D}$ is established. The uniqueness part of the theorem is demonstrated in the lemma that follows. ☐

Lemma 12 (Poincaré)**.** *Let $\Omega \subset \mathbb{C}$ be an open set, and let $\alpha \in \Omega$. Assume that $\phi : \Omega \to \mathbb{D}$ and $\psi : \Omega \to \mathbb{D}$ are biholomorphic mappings onto $\mathbb{D}$ such that $\phi(\alpha) = \psi(\alpha) = 0$ and that both $\phi'(\alpha)$ and $\psi'(\alpha)$ are positive real numbers. Then $\phi = \psi$.*

Proof. Since $\phi \circ \psi^{-1} \in \operatorname{Aut} \mathbb{D}$ and $\phi(\psi^{-1}(0)) = 0$, we have $\phi \circ \psi^{-1}$ is a rotation around 0. In other words, $\phi(z) = \lambda\psi(z)$ with some $\lambda \in \partial\mathbb{D}$. Therefore $\phi'(\alpha) = \lambda\psi'(\alpha)$, and so $\lambda = 1$. ☐

5.1. **Koebe's square-root trick.** Recall from the proof of Theorem 11 that Φ is the set of holomorphic injections $\varphi : \Omega \to \mathbb{D}$ that satisfy $\varphi(\alpha) = 0$. In this subsection we establish the nonemptiness of Φ, by explicitly constructing an element of Φ.

The complexity of the construction depends on Ω. If Ω is bounded, a translation (that sends α to 0) followed by a dilation will do. If Ω is unbounded and if there is an open disk $D_r(c) \subset \mathbb{C} \setminus \Omega$, then the transformation $z \mapsto \frac{1}{z-c}$ will render Ω bounded, and so the problem reduces to the previous case.

In general, since $\Omega \neq \mathbb{C}$ there is $c \in \mathbb{C} \setminus \Omega$. Let $f(z) = z - c$ and let $g \in \mathcal{O}(\Omega)$ be such that $g(\cdot)^2 = f(\cdot)$. The existence of g is guaranteed by the square-root property of Ω. We have

$$g(z_1) = g(z_2) \qquad \Rightarrow \qquad z_1 - c = g(z_1)^2 = g(z_2)^2 = z_2 - c \qquad \Rightarrow \qquad z_1 = z_2,$$

meaning that g is injective. On the other hand, the same argument gives

$$g(z_1) = -g(z_2) \qquad \Rightarrow \qquad z_1 = z_2,$$

meaning that if $0 \neq w \in g(\Omega)$, then $-w \notin g(\Omega)$. Now $g(\Omega)$ is an open set, so there exists an open disk $D \subseteq g(\Omega)$ with $D \not\ni 0$. Then the disk $-D = \{-w : w \in D\}$ is in the complement of $g(\Omega)$, i.e., $-D \subset \mathbb{C} \setminus g(\Omega)$, which reduces the problem to the previous case. Let us state what we have proved here as a lemma.

Lemma 13. *Let $\Omega \subsetneq \mathbb{C}$ be a connected, open set having the square-root property, and let $\alpha \in \Omega$. Then there is a holomorphic injection $\varphi : \Omega \to \mathbb{D}$ that satisfies $\varphi(\alpha) = 0$.*

5.2. **Montel's theorem.** In this subsection, we discuss compactness properties of Φ. First we recall the celebrated Arzelà-Ascoli theorem, in a form convenient for our purposes.

Theorem 14 (Arzelà-Ascoli)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $f_n : \Omega \to \mathbb{C}$ be a sequence that is locally equicontinuous and locally equibounded. Then there is a subsequence of $\{f_n\}$ that converges locally uniformly.*

That the sequence $\{f_n\}$ is *locally equibounded* means that for any compact set $K \subset \Omega$ one has $\sup_n \|f_n\|_K < \infty$. Similarly, that the sequence $\{f_n\}$ is *locally equicontinuous* means that for any compact set $K \subset \Omega$ the sequence $\{f_n\}$ is (uniformly) equicontinuous on K. It turns out that if $\{f_n\}$ is a sequence of holomorphic functions, then the local equicontinuity condition can be dropped from the Arzelà-Ascoli theorem.

Theorem 15 (Montel)**.** *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $\{f_n\} \subset \mathcal{O}(\Omega)$ be a locally equibounded sequence. Then there is a subsequence of $\{f_n\}$ that converges locally uniformly.*

Proof. In view of the Arzelà-Ascoli theorem, it suffices to show local equicontinuity of $\{f_n\}$. We will prove here that $\{f_n\}$ is equicontinuous on any closed disk $\overline{D} \subset \Omega$, and the general case follows by a covering argument. Let $D = D_\rho(c)$ and $D' = D_{\rho+\delta}(c)$ be two concentric disks such that $\overline{D'} \subset \Omega$ and $\delta > 0$. Then the Cauchy estimates give

$$|f_n'(z)| \leq \frac{1}{\delta} \max_{\partial D_\delta(z)} |f_n|, \quad \text{for} \quad z \in D, \qquad \text{and so} \qquad \|f_n'\|_D \leq \frac{1}{\delta}\|f_n\|_{D'}.$$

For $z, w \in D$, we have

$$|f_n(z) - f_n(w)| = |\langle f_n', [zw]\rangle| \leq |z-w| \cdot \|f_n'\|_{[zw]} \leq |z-w| \cdot \frac{1}{\delta}\|f_n\|_{D'}.$$

Since $\|f_n\|_{D'}$ is bounded uniformly in n, the sequence $\{f_n\}$ is equicontinuous on D. □

5.3. **Carathéodory-Koebe expansion map.** Recall from the proof of Theorem 11 that Φ is the set of holomorphic injections $\varphi : \Omega \to \mathbb{D}$ that satisfy $\varphi(\alpha) = 0$, and that $\phi \in \Phi$ is an element of Φ satisfying $|\phi(\beta)| = \sup\{|\varphi(\beta)| : \varphi \in \Phi\}$, where $\alpha, \beta \in \Omega$ and $\alpha \neq \beta$. The goal of this subsection is to show that $\phi : \Omega \to \mathbb{D}$ is surjective.

Let $\Sigma = \phi(\Omega)$. We say that a holomorphic injection $\kappa : \Sigma \to \mathbb{D}$ is an *expansion* if $\kappa(0) = 0$ and $|\kappa(z)| > |z|$ for any $z \in \Sigma \setminus \{0\}$. Suppose that there exists an expansion $\kappa : \Sigma \to \mathbb{D}$. Then we have $\kappa \circ \phi \in \Phi$, and

$$|[\kappa \circ \phi](\beta)| = |\kappa(\phi(\beta))| > |\phi(\beta)|,$$

so from the maximality of $|\phi(\beta)|$ we conclude that there does not exist an expansion $\kappa : \Sigma \to \mathbb{D}$. But we show that there does exist an expansion $\kappa : \Sigma \to \mathbb{D}$ if $\Sigma \neq \mathbb{D}$, implying that $\Sigma = \mathbb{D}$.

Lemma 16. *Let $\Sigma \subsetneq \mathbb{D}$ be a connected, open set having the square-root property, with $0 \in \Sigma$. Then there exists an expansion $\kappa : \Sigma \to \mathbb{D}$.*

Proof. For $\alpha \in \mathbb{D}$, let us define

$$\varphi_\alpha(z) = \frac{z-\alpha}{\bar{\alpha}z - 1} = -B_\alpha(z),$$

where $B_\alpha \in \operatorname{Aut}\mathbb{D}$ is the Blaschke factor defined in §4. So we have $\varphi_\alpha \in \operatorname{Aut}\mathbb{D}$, and $\varphi_\alpha^{-1} = \varphi_\alpha$. We also define the following maps sending $\mathbb{D}$ into $\mathbb{D}$:

$$j(z) = z^2, \qquad \text{and} \qquad \psi_\alpha = \varphi_{\alpha^2} \circ j \circ \varphi_\alpha,$$

and so in particular we have

$$\psi_\alpha(0) = \varphi_{\alpha^2}(\varphi_\alpha(0)^2) = \varphi_{\alpha^2}(\alpha^2) = 0.$$

Moreover, since $j : \mathbb{D} \to \mathbb{D}$ is not an automorphism, ψ_α is not a rotation, hence by Schwarz's lemma we have

$$|\psi_\alpha(z)| < |z|, \qquad \text{for all} \qquad z \in \mathbb{D} \setminus \{0\}.$$

We see that ψ_α is a contraction as opposed to an expansion, but in what follows we will construct its inverse on Σ which is an expansion.

Let $\alpha \in \mathbb{D}$ be such that $\alpha^2 \notin \Sigma$. This is possible since $\Sigma \subsetneq \mathbb{D}$. Then $\varphi_{\alpha^2} : \Sigma \to \mathbb{D}$ is zero-free and satisfies $\varphi_{\alpha^2}(0) = \alpha^2$. By the square-root property, there is $\eta \in \mathcal{O}(\Sigma)$ such that $\eta(z)^2 = \varphi_{\alpha^2}(z)$ for all $z \in \Sigma$, or equivalently, $\varphi_{\alpha^2} = j \circ \eta$ on Σ, and that $\eta(0) = \alpha$. Since $\varphi_{\alpha^2}(\Sigma) \subset \mathbb{D}$, we have $\eta(\Sigma) \subset \mathbb{D}$. Now we define $\kappa = \varphi_\alpha \circ \eta$. Then $\kappa(\Sigma) \subset \mathbb{D}$, $\kappa(0) = 0$, and

$$\psi_\alpha \circ \kappa = \varphi_{\alpha^2} \circ j \circ \varphi_\alpha \circ \varphi_\alpha \circ \eta = \varphi_{\alpha^2} \circ j \circ \eta = \varphi_{\alpha^2} \circ \varphi_{\alpha^2} = \mathrm{id}_\Sigma,$$

where $\mathrm{id}_\Sigma : \Sigma \to \Sigma$ is the identity mapping on Σ. This implies that $\kappa : \Sigma \to \mathbb{D}$ is injective, and in particular that if $z \neq 0$ then $\kappa(z) \neq 0$. Finally, we infer

$$|z| = |\psi_\alpha(\kappa(z))| < |\kappa(z)|, \qquad \text{for all} \qquad z \in \Sigma \setminus \{0\},$$

from the contraction property of ψ_α, which completes the proof. □