MATH 566 LECTURE NOTES 4: ISOLATED SINGULARITIES AND THE RESIDUE THEOREM

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1. Functions holomorphic on an annulus

Let $A = D_R \setminus \overline{D}_r$ be an annulus centered at 0 with $0 < r < R < \infty$, and consider a function $f \in \mathcal{O}(\overline{A})$. Then the Cauchy integral formula gives

$$f(\zeta) = \frac{1}{2\pi i}\langle K_\zeta f, \partial A\rangle = \frac{1}{2\pi i}\langle K_\zeta f, \partial D_R\rangle - \frac{1}{2\pi i}\langle K_\zeta f, \partial D_r\rangle, \qquad \text{for} \quad \zeta \in A.$$

Proceeding as in the proof of the Cauchy-Taylor theorem, the first term on the right hand side can be written as

$$f^+(\zeta) := \frac{1}{2\pi i}\langle K_\zeta f, \partial D_R\rangle = \frac{1}{2\pi i}\sum_{n=0}^{\infty} \zeta^n \int_{\partial D_R} \frac{f(z)\mathrm{d}z}{z^{n+1}}, \qquad \text{for} \quad \zeta \in A,$$

with the series converging locally normally in D_R, so in particular $f^+ \in \mathcal{O}(D_R)$. For the other term, we have

$$f^-(\zeta) := -\frac{1}{2\pi i}\langle K_\zeta f, \partial D_r\rangle = \frac{1}{2\pi i}\int_{\partial D_r} \frac{f(z)\mathrm{d}z}{\zeta - z},$$

and similarly to the proof of the Cauchy-Taylor theorem, this can be rewritten as

$$f^-(\zeta) = \frac{1}{2\pi i}\int_{\partial D_r}\left(\sum_{n=0}^{\infty}\frac{f(z)z^n}{\zeta^{n+1}}\right)\mathrm{d}z, \tag{1}$$

where we have used

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\cdot\frac{1}{1 - z/\zeta} = \frac{1}{\zeta}\left(1 + \frac{z}{\zeta} + \ldots\right) = \sum_{n=0}^{\infty}\frac{z^n}{\zeta^{n+1}}.$$

Each term in the series under integral in (1) can be estimated as

$$\left|\frac{f(z)z^n}{\zeta^{n+1}}\right| \leq \frac{\|f\|_A}{|\zeta|}\cdot\left(\frac{r}{|\zeta|}\right)^n,$$

so as a function of z, the series converges uniformly on ∂D_r, as long as $\zeta \in \mathbb{C}\setminus\overline{D}_r$. Therefore we can interchange the integral with the sum, resulting in

$$f^-(\zeta) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{\zeta^{n+1}}\int_{\partial D_r} f(z)z^n\mathrm{d}z = \frac{1}{2\pi i}\sum_{n=-1}^{-\infty}\zeta^n\int_{\partial D_r}\frac{f(z)\mathrm{d}z}{z^{n+1}}.$$

Now the individual term of the series satisfies

$$\left|\frac{1}{\zeta^{n+1}}\int_{\partial D_r} f(z)z^n\mathrm{d}z\right| \leq 2\pi\|f\|_A\left(\frac{r}{|\zeta|}\right)^{n+1}, \qquad n \geq 0,$$

implying that the series converges locally normally in $\mathbb{C}\setminus\overline{D}_r$.

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Finally, we note that since f is holomorphic on a neighbourhood of A, the integrals in the series expansions of f^+ and f^- stay constant if we change the contours ∂D_r and ∂D_R by, say, ∂D_ρ with some $\rho \in [r, R]$, and in particular we can write the two series together as

$$f(\zeta) = f^+(\zeta) + f^-(\zeta) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \zeta^n \int_{\partial D_\rho} \frac{f(z)\mathrm{d}z}{z^{n+1}}.$$

This series is called the *Laurent series* of f, and the decomposition $f = f^+ + f^-$ is called the *Laurent decomposition* of f. In the Laurent decomposition, f^- is said to be the *principal part* of f, and f^+ the *regular part* of f. We summarize this discussion in the following theorem.

Theorem 1. *Let $A = D_R(c) \setminus \overline{D}_r(c)$ be an annulus centered at $c \in \mathbb{C}$ with $0 \leq r < R \leq \infty$. Then any $f \in \mathcal{O}(A)$ has a unique decomposition $f = f^+ + f^-$ such that $f^+ \in \mathcal{O}(D_R(c))$ and $f^- \in \mathcal{O}(\mathbb{C} \setminus \overline{D}_r(c))$ with $f^-(z) \to 0$ as $|z| \to \infty$. Moreover, the series*

$$f^+(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - c)^n, \qquad \text{and} \qquad f^-(\zeta) = \sum_{n=-1}^{-\infty} a_n(\zeta - c)^n,$$

converge respectively in $D_R(c)$ and in $\mathbb{C} \setminus \overline{D}_r(c)$ both locally normally, where the coefficients are given by

$$a_n = \frac{1}{2\pi i} \langle K_c^{n+1} f, \partial D_\rho(c) \rangle \equiv \frac{1}{2\pi i} \int_{\partial D_\rho(c)} \frac{f(z)\mathrm{d}z}{(z-c)^{n+1}}, \qquad r < \rho < R.$$

Proof. The existence of the Laurent decomposition, and the convergence of the two series are demonstrated above. For uniqueness, let $f = g^+ + g^-$ be another such decomposition of f. Then we have $f^+ - g^+ = g^- - f^-$ on A, and so h defined as $h = f^+ - g^+$ on D_R and $h = g^- - f^-$ on $\mathbb{C} \setminus \overline{D}_r$ is an entire function. But $g^- - f^-$ goes to 0 at ∞, hence by Liouville's theorem $h \equiv 0$. $\square$

2. Isolated singularities

We call the set $D_r^\times(c) = \{z \in \mathbb{C} : 0 < |z - c| < r\}$ the *punctured disk* centred at c with radius r. If $f \in \mathcal{O}(D_r^\times(c))$ then c is called an *isolated singularity* of f, and we have the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - c)^n, \qquad z \in D_r^\times(c).$$

Let us introduce the notation $\mathrm{Ord}(f, c) = \inf\{n \in \mathbb{Z} : a_n \neq 0\}$, with $\mathrm{Ord}(f, c) = \infty$ if $f \equiv 0$ on a neighbourhood of c, and $\mathrm{Ord}(f, c) = -\infty$ if there is infinitely many $a_n \neq 0$ with $n < 0$. Note that if $\mathrm{Ord}(f, c) \geq 0$ then upon defining $f(c) = a_0$, one has $f \in \mathcal{O}(D_r(c))$. This process of extending the definition of f to include c in the domain of f so that the resulting function is holomorphic, is called *removing the singularity at c*. In what follows, we will often implicitly assume that all singularities that can be removed have been removed, and so in particular function values at "removable singularity" points will be discussed without ever mentioning the process of removing those singularities. For example, note that $\mathrm{Ord}(f, c) > 0$ is equivalent to saying that c is a *zero* of order $\mathrm{Ord}(f, c)$. We will see that one cannot remove the singularity at c if $\mathrm{Ord}(f, c) < 0$.

Definition 2. In the above setting, c is called

- a *removable singularity* if $\mathrm{Ord}(f, c) \geq 0$;
- a *pole* of order $N := -\mathrm{Ord}(f, c)$ if $0 > \mathrm{Ord}(f, c) > -\infty$;
- an *essential singularity* if $\mathrm{Ord}(f, c) = -\infty$.

If $\mathrm{Ord}(f, c) = -1$ then we say c is a *simple* pole, and likewise a *double* pole has $\mathrm{Ord}(f, c) = -2$.

Sometimes very useful is the trivial fact that the three cases in the preceding definition are mutually exclusive and exhaust all possibilities. It is clear that if f has a removable singularity at c then f is bounded on a neighbourhood of c. The converse statement is also true.

Theorem 3 (Riemann's removable singularity theorem)**.** *If $f \in \mathcal{O}(D_r^\times(c))$ and f is bounded on $D_r^\times(c)$, then c is a removable singularity.*

Proof. Assume $c = 0$. The Laurent series coefficients of f are given by

$$a_n = \frac{1}{2\pi i} \int_{\partial D_\rho} \frac{f(z) \mathrm{d}z}{z^{n+1}},$$

where $\rho \in (0, r)$. These can be estimated as

$$|a_n| \leq \frac{1}{2\pi} \cdot \frac{\|f\|_{D_r^\times}}{\rho^{n+1}} \cdot 2\pi\rho = \rho^{-n} \|f\|_{D_r^\times} \to 0 \qquad \text{as} \quad \rho \to 0, \quad \text{for} \quad n < 0,$$

which shows that $a_n = 0$ for all $n < 0$. □

Recall that a function is holomorphic at a point means that the function is holomorphic on a neighbourhood of that point.

Theorem 4 (Characterization of poles)**.** *If $f \in \mathcal{O}(D_r^\times(c))$ then the followings are equivalent:*
(a) c is a pole of f of order N.
(b) $g(z) = (z - c)^N f(z)$ is holomorphic at c with $g(c) \neq 0$.
(c) $1/f$ is holomorphic at c with zero of order N at c.

Proof. If (a) is satisfied then from the Laurent series expansion of f we have

$$g(z) = (z - c)^N f(z) = \sum_{n=0}^{\infty} a_{n-N} (z - c)^n, \qquad \text{with} \quad g(c) = a_{-N} \neq 0,$$

implying (b).

If (b) is true, then we have $\dfrac{1}{f(z)} = \dfrac{(z-c)^N}{g(z)}$ and g is holomorphic and nonzero at c, so $\dfrac{1}{f}$ is holomorphic and has zero of order N at c.

Now (c) implies that $\dfrac{1}{f(z)} = \dfrac{(z-c)^N}{h(z)}$ with some h holomorphic and nonzero at c. Then by inverting this relation and expanding h in its Taylor series around c, we get (a). □

From (c) of this theorem it follows that $|f|$ approaches ∞ near the poles. The converse can also be proved with a little effort.

Corollary 5. *If $f \in \mathcal{O}(D_r^\times(c))$ and $|f(z)| \to \infty$ as $z \to c$ then c is a pole of f.*

Proof. Firstly, we infer that f is zero-free on $D_\varepsilon^\times(c)$ for some $\varepsilon > 0$. Hence $1/f \in \mathcal{O}(D_\varepsilon^\times(c))$ and since it is also bounded, by the removable singularity theorem we have $1/f \in \mathcal{O}(D_\varepsilon(c))$. Moreover, $1/f$ is zero at c, which therefore is a pole of f by (c) of the preceding theorem. □

Theorem 6 (Casorati-Weierstrass theorem)**.** *Let $f \in \mathcal{O}(D_r^\times(c))$ and let c be an essential singularity of f. Then for any $\alpha \in \mathbb{C}$, there is a sequence $z_n \to c$ such that $f(z_n) \to \alpha$. In other words, $f(D_\varepsilon^\times(c))$ is dense in $\mathbb{C}$ for any $\varepsilon > 0$.*

Proof. Suppose for the sake of contradiction that there are $\alpha \in \mathbb{C}$ and positive numbers δ, and ε such that $|f(z) - \alpha| \geq \delta$ for any $z \in D_\varepsilon^\times(c)$. Then $h(z) = 1/(f(z) - \alpha)$ is bounded in $D_\varepsilon^\times(c)$, hence holomorphic on $D_\varepsilon(c)$. Unravelling, we have $f(z) = \alpha + 1/h(z)$, and since h is not identically zero, $\mathrm{Ord}(f, c) > -\infty$. □

The converse of the Casorati-Weierstrass theorem also holds, because if f has the property as in the conclusion of this theorem at c, then c is neither a removable singularity (in which case f would have to be bounded) nor a pole (in which case $|f|$ would have to have the limit ∞), so it must be an essential singularity. A similar exclusion of both a removable and an essential singularity gives an alternative proof of Corollary 5.

3. Residues and indices

Let us recall the definition of residue from the previous set of notes.

Definition 7. If $f \in \mathcal{O}(D_r^\times(c))$ with $r > 0$, then with $\varepsilon \in (0, r)$

$$\operatorname{Res}(f, c) = \frac{1}{2\pi i}\langle f, \partial D_\varepsilon(c)\rangle \equiv \frac{1}{2\pi i}\int_{\partial D_\varepsilon(c)} f(z)\mathrm{d}z,$$

is called the *residue* of f at c. Note that the residue does not depend on the value of $\varepsilon \in (0, r)$, so in particular one could take the limit $\varepsilon \to 0$.

We have a new tool to examine functions such as f in this definition: The Laurent series expansion. So employing this new tool right away we have the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n, \qquad z \in D_r^\times(c),$$

which in particular converges uniformly on $\partial D_\varepsilon(c)$ for $\varepsilon \in (0, r)$. This means that integration of f over $\partial D_\varepsilon(c)$ can be interchanged with the series summation, giving

$$\operatorname{Res}(f, c) = \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} a_n \int_{\partial D_\varepsilon(c)} (z - c)^n \mathrm{d}z = a_{-1},$$

where the integral of $(z - c)^n$ vanishes except for $n = -1$ because of integrability. We collect some useful properties of residue in the following lemma.

Lemma 8. *Let $c \in \mathbb{C}$, and let $f, g \in \mathcal{O}(D_r^\times(c))$ with $r > 0$.*

(a) If a_{-1} is the (-1)st coefficient in the Laurent series of f around c, then

$$\operatorname{Res}(f, c) = a_{-1}.$$

(b) If $\operatorname{Ord}(f, c) \geq 0$ (removable singularity) then

$$\operatorname{Res}(f, c) = 0.$$

(c) If $\operatorname{Ord}(f, c) \geq -1$ (simple pole at worst) then

$$\operatorname{Res}(f, c) = \lim_{z\to c}(z - c)f(z).$$

(d) If $\operatorname{Ord}(f, c) = 0$ and $\operatorname{Ord}(g, c) = 1$, then

$$\operatorname{Res}(f/g, c) = \frac{f(c)}{g'(c)}.$$

(e) $\operatorname{Res}(\cdot, c)$ is $\mathbb{C}$-linear in $\mathcal{O}(D_r^\times(c))$.

Proof. Part (a) is proven above, and (b), (c) and (e) are obvious. Then (d) follows from (c) as

$$\operatorname{Res}(f/g, c) = \lim_{z\to c}\frac{(z - c)f(z)}{g(z)} = \lim_{z\to c}\frac{(z - c)f(z)}{g(z) - g(c)} = \frac{f(c)}{g'(c)},$$

since $g(c) = 0$ and $g'(c) \neq 0$ by $\operatorname{Ord}(g, c) = 1$. □

Let us recall also the definition of index.

Definition 9. For a cycle $\gamma \in Z_1(\mathbb{C})$ that does not pass through $c \in \mathbb{C}$, the *index* (or the *winding number*) of γ with respect to c is defined to be

$$\mathrm{Ind}(\gamma, c) = \frac{1}{2\pi i}\langle K_c, \gamma\rangle \equiv \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-c}.$$

The following theorem was derived in the previous set of notes, by using a certain result on the homology group of simple domains. We give here a self-contained proof.

Theorem 10 (Residue theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $z_1, \ldots, z_m \in \Omega$. Suppose that $\gamma \in B_1(\Omega)$ is a null-homologous cycle that does not pass through any of the points $z_1, \ldots, z_m$. Then we have*

$$\langle f, \gamma\rangle = 2\pi i \sum_{j=1}^m \mathrm{Ind}(\gamma, z_j)\,\mathrm{Res}(f, z_j), \qquad \textit{for} \qquad f \in \mathcal{O}(\Omega \setminus \{z_1, \ldots, z_m\}).$$

Proof. Let $\varepsilon > 0$ be such that the disks $D_\varepsilon(z_j)$ are pairwise disjoint and disjoint from the boundary $\partial\Omega$. Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_{j,n}(z-z_j)^n, \qquad z \in D_\varepsilon^\times(z_j),$$

be the Laurent series of f around z_j, and let

$$f_j^-(z) = \sum_{n=-1}^{-\infty} a_{j,n}(z-z_j)^n, \qquad z \in D_\varepsilon^\times(z_j),$$

be the principal part of f at z_j. We have $f_j^- \in \mathcal{O}(\mathbb{C}\setminus\{z_j\})$ and $f - f_j^- \in \mathcal{O}(D_\varepsilon(z_j))$, so that $f - f_1^- - \ldots - f_m^- \in \mathcal{O}(D_\varepsilon(z_j))$ for each j. We also have $f - f_1^- - \ldots - f_m^- \in \mathcal{O}(\Omega\setminus\{z_1, \ldots, z_m\})$, which then implies that $f - f_1^- - \ldots - f_m^- \in \mathcal{O}(\Omega)$. Since γ is null-homologous in Ω, Cauchy's theorem gives

$$\langle f, \gamma\rangle = \sum_{j=1}^m \langle f_j^-, \gamma\rangle.$$

Now taking into account the compactness of $|\gamma|$, we infer

$$\langle f_j^-, \gamma\rangle = \int_\gamma \sum_{n=-1}^{-\infty} a_{j,n}(z-z_j)^n \mathrm{d}z = \sum_{n=-1}^{-\infty} a_{j,n}\int_\gamma (z-z_j)^n \mathrm{d}z = 2\pi i a_{j,-1}\mathrm{Ind}(\gamma, z_j),$$

and recalling that $a_{j,-1} = \mathrm{Res}(f, z_j)$ completes the proof. $\square$

From the homological derivation of the index formula, $\mathrm{Ind}(\gamma, z)$ should correspond to the number of times γ wraps (counterclockwise) around z. Yet we do not even know if $\mathrm{Ind}(\gamma, z)$ would be an integer for a closed curve γ, without resorting to the topological result that γ is homologous in $\mathbb{C}\setminus\{z\}$ to a unique integer multiple of an oriented circular loop around z. We will justify below in a self-contained manner that $\mathrm{Ind}(\gamma, z)$ indeed makes precise the intuitive notion of the winding number of γ around z.

Let $\gamma : [0, 1] \to \mathbb{C}$ be a piecewise differentiable loop that does not pass through $a \in \mathbb{C}$. With $t \in [0, 1]$, we partition the interval $[0, t]$ as $0 = t_0 < \ldots < t_n = t$ such that $\sigma_j = \gamma|_{[t_{j-1}, t_j]} \subset D_j$ where $D_j \subset \mathbb{C}\setminus\{a\}$ are open disks. On each D_j, there exists a logarithm of $z - a$, that is

$$\log_j(z-a) = \log|z-a| + i\arg_j(z-a),$$

where $z \mapsto \arg_j(z-a)$ is an argument function on D_j. Since $\log_j'(z-a) = \frac{1}{z-a}$, we have

$$\int_{\sigma_j} \frac{\mathrm{d}z}{z-a} = \log_j \frac{\gamma(t_j)-a}{\gamma(t_{j-1})-a}, \tag{2}$$

and by exponentiating,

$$\exp\langle K_a, \sigma_j\rangle \equiv \exp \int_{\sigma_j} \frac{\mathrm{d}z}{z-a} = \frac{\gamma(t_j)-a}{\gamma(t_{j-1})-a}.$$

For $\sigma = \sigma_1 + \ldots + \sigma_n \equiv \gamma|_{[0,t]}$, this gives

$$\exp\langle K_a, \sigma\rangle = \exp \sum_{j=1}^{n} \langle K_a, \sigma_j\rangle = \prod_{j=1}^{n} \exp\langle K_a, \sigma_j\rangle = \frac{\gamma(t_n)-a}{\gamma(t_0)-a}, \tag{3}$$

and since γ is a loop, we have $\exp\langle K_a, \gamma\rangle = 1$, meaning that $\langle K_a, \gamma\rangle = 2\pi i k$ for some $k \in \mathbb{Z}$. This concludes that $\mathrm{Ind}(\gamma, a)$ is an integer whenever γ is a piecewise differentiable loop that does not pass through $a \in \mathbb{C}$. The same conclusion also holds for piecewise differentiable cycles, since any such a cycle is homologous in $\mathbb{C} \setminus \{a\}$ to a linear combination (with integer coefficients) of polygonal loops, and K_a is holomorphic in $\mathbb{C} \setminus \{a\}$.

To get a better idea of how $\mathrm{Ind}(\gamma, a)$ relates to the number of times γ winds around a, let us write (2) in its real and imaginary parts

$$\langle K_a, \sigma_j\rangle = \log \frac{|\gamma(t_j)-a|}{|\gamma(t_{j-1})-a|} + i\left(\arg_j(\gamma(t_j)-a) - \arg_j(\gamma(t_{j-1})-a)\right).$$

Note that although there are many choices of the argument function, the argument increment

$$\arg_j(\gamma(t_j)-a) - \arg_j(\gamma(t_{j-1})-a)$$

does not depend on this choice. Recalling that $\sigma = \sigma_1 + \ldots + \sigma_n \equiv \gamma|_{[0,t]}$, we have

$$\eta(t) := \langle K_a, \sigma\rangle = \log \frac{|\gamma(t)-a|}{|\gamma(0)-a|} + i\sum_{j=1}^{n}\left(\arg_j(\gamma(t_j)-a) - \arg_j(\gamma(t_{j-1})-a)\right).$$

We know that $\eta(0) = 0$ and $\eta(1) = 2\pi i k \in 2\pi i\mathbb{Z}$, with $k = \mathrm{Ind}(\gamma, a)$. Moreover, from the preceding formula it is clear that as t increases from 0 to 1, the imaginary part (or the vertical motion) of $\eta(t)$ records exactly the angular movement of the vector $\gamma(t) - a$. Combining (3) with the definition of η, we infer that $\gamma(t) = a + (\gamma(0) - a)\exp \eta(t)$ for $t \in [0, 1]$, or $\gamma = a + (\gamma(0) - a)\exp\circ\eta$. On the other hand, η is homotopic (relative to fixed endpoints) to the vertical line segment $[0, 2\pi i k]$, and $a + (\gamma(0) - a)\exp\circ[0, 2\pi i k]$ is simply a circular path that winds k times around a. Finally, this implies that for $\varepsilon > 0$, γ is freely homotopic to the k-fold circle $k \cdot \partial D_\varepsilon(a)$ in $\mathbb{C} \setminus \{a\}$.

In the following theorem, we collect some simple properties of the index function. If γ is a piecewise differentiable cycle, we define $\mathrm{Int}\gamma = \{z \in \mathbb{C} : \mathrm{Ind}(\gamma, z) \neq 0\}$ to be the *interior* of γ, and $\mathrm{Ext}\gamma = \{z \in \mathbb{C} : \mathrm{Ind}(\gamma, z) = 0\}$ to be the *exterior* of γ.

Theorem 11. *Let $\gamma \in Z_1(\mathbb{C})$ be a piecewise differentiable cycle. Then we have*
(a) $\mathrm{Ind}(\gamma, a) \in \mathbb{Z}$ *whenever $a \notin |\gamma|$.*
(b) $\mathrm{Ind}(\gamma, \cdot)$ *is locally constant in $\mathbb{C} \setminus |\gamma|$.*
(c) $\mathbb{C} = \mathrm{Int}\gamma \cup |\gamma| \cup \mathrm{Ext}\gamma$ *and the decomposition is disjoint.*
(d) If γ is null-homologous in Ω, then $\mathrm{Int}\gamma \subset \Omega$.
(e) If $|\gamma| \subset \Omega$ with Ω simply connected, then $\mathrm{Int}\gamma \subset \Omega$ *and* $\mathbb{C} \setminus \Omega \subset \mathrm{Ext}\gamma$.
(f) $\mathrm{Int}\gamma$ *is bounded, and* $\mathrm{Ext}\gamma$ *is nonempty and unbounded.*

Proof. Part (a) is demonstrated above, and (b) follows from continuity. Then (c) is true by definition, and (d) follows from the fact that if $a \in \mathbb{C} \setminus \Omega$ then $K_a \in \mathcal{O}(\Omega)$ and so $\langle K_a, \gamma\rangle = 0$. In a simply connected domain every cycle is null-homologous, hence the first claim of (e) follows from (d), and then the second claim follows from (c). Now, since $|\gamma|$ is bounded there is a large disk D such that $|\gamma| \subset D$, and by applying (e) we have (f). □

Making use of these properties, we can strengthen the residue theorem so as to allow infinitely many isolated singularity points.

Theorem 12 (Residue theorem, stronger version)**.** *Let $\Omega \subset \mathbb{C}$ be open, and let $K \subset \Omega$ be a discrete set. Suppose that $\gamma \in B_1(\Omega)$ is a null-homologous cycle that does not pass through any of the points in K. Then we have*

$$\langle f, \gamma \rangle = 2\pi i \sum_{c \in K} \mathrm{Ind}(\gamma, c) \, \mathrm{Res}(f, c), \qquad \text{for} \qquad f \in \mathcal{O}(\Omega \setminus K),$$

where there are only finitely many nonzero summands in the sum on the right hand side.

Proof. Since $|\gamma|$ is compact, there is an open bounded U such that $\overline{U} \subset \Omega$ and $|\gamma| \subset U$. Then $U \cap K$ is finite by the discreteness of K, and $\mathrm{Ind}(\gamma, z) = 0$ for $z \in \mathbb{C} \setminus U$ by the local constancy of $\mathrm{Ind}(\gamma, \cdot)$. Now we apply Theorem 10 to U to complete the proof. ☐

4. The argument principle

Definition 13. Let $\Omega \subseteq \mathbb{C}$ be open, and let $K \subset \Omega$ be a discrete set. Then $f : \Omega \setminus K \to \mathbb{C}$ is called *meromorphic* on M if $f \in \mathcal{O}(M \setminus K)$ and f has a pole at each point of K. The set of meromorphic functions on Ω is denoted by $\mathcal{M}(\Omega)$.

The argument principle is a result that gives a way to count the number of zeroes and poles in a given region. Note that the counting takes into account the multiplicities.

Theorem 14 (Argument principle)**.** *Let $f \in \mathcal{M}(\Omega)$ be nonzero, and let γ be a null-homologous cycle in Ω. Suppose also that $\mathrm{Ord}(f, z) = 0$ for all $z \in |\gamma|$. Then we have*

$$2\pi i \sum_{c \in \mathrm{Int}\gamma} \mathrm{Ind}(\gamma, c) \mathrm{Ord}(f, c) = \langle f'/f, \gamma \rangle = \mathrm{Ind}(f \circ \gamma, 0),$$

where there are only finitely many nonzero summands in the sum on the left hand side.

Proof. Setting $N = \mathrm{Ord}(f, c)$, we can write $f(z) = (z - c)^N h(z)$ on a neighbourhood of c with h holomorphic and $h(c) \neq 0$. So we calculate, on a punctured neighbourhood of c,

$$f'(z) = N(z - c)^{N-1} h(z) + (z - c)^N h'(z), \qquad \text{and so} \qquad \frac{f'(z)}{f(z)} = \frac{N}{z - c} + \frac{h'(z)}{h(z)},$$

which gives $\mathrm{Res}(f'/f, c) = N$. Now the residue theorem completes the proof. ☐

The argument principle says that the number of zeroes and poles of f depends on f continuously (under certain conditions). So being an integer, this number must be stable under finite but not too large perturbations of f.

Theorem 15 (Rouché's theorem)**.** *Let $f, g \in \mathcal{M}(\Omega)$, and let γ be a null-homologous cycle in Ω. Suppose that $|f - g| < |g| < \infty$ on $|\gamma|$. Then we have*

$$\sum_{c \in \mathrm{Int}\gamma} \mathrm{Ind}(\gamma, c) \mathrm{Ord}(f, c) = \sum_{c \in \mathrm{Int}\gamma} \mathrm{Ind}(\gamma, c) \mathrm{Ord}(g, c).$$

Proof. Let $h_s = g + s(f - g)$ for $s \in [0, 1]$. Then we have

$$|h_s| \leq |g| + |f - g| < 2|g| < \infty, \qquad \text{and} \qquad |h_s| \geq |g| - |f - g| > 0, \qquad \text{on} \quad |\gamma|.$$

We apply the argument principle to h_s and define

$$\eta(s) = \sum_{c \in \mathrm{Int}\gamma} \mathrm{Ind}(\gamma, c) \mathrm{Ord}(h_s, c) = \frac{1}{2\pi i} \left\langle \frac{g' + s(f' - g')}{g + s(f - g)}, \gamma \right\rangle, \qquad s \in [0, 1].$$

It is obvious that η is continuous and takes integer values, so we have $\eta(0) = \eta(1)$. Since $h_0 = g$ and $h_1 = f$, this establishes the proof. ☐

Remark 16. This theorem is still valid if the condition $|f-g|<|g|<\infty$ is replaced by the weaker condition $|f-g|<|f+g|<\infty$.

As an immediate application let us prove a fixed point theorem.

Corollary 17. *If $\phi \in \mathcal{O}(\overline{\mathbb{D}})$ and $\phi(\partial\mathbb{D}) \subset \mathbb{D}$ then there is a unique fixed point of ϕ in $\mathbb{D}$, i.e., there is a unique $z \in \mathbb{D}$ such that $\phi(z)=z$.*

Proof. Take $f(z)=z-\phi(z)$ and $g(z)=z$ in Hurwitz's theorem. We have $|f-g|=|\phi|<1=|g|$ on $\partial\mathbb{D}$, and so $z-\phi(z)$ and z have the same number of zeroes in $\mathbb{D}$. $\square$

Rouché's theorem implies that the zeroes of (locally) uniformly converging holomorphic functions must condense in a certain sense.

Theorem 18 (Hurwitz's theorem)**.** *Let $\{f_n\} \subset \mathcal{O}(\Omega)$ be a sequence such that $f_n \to f \in \mathcal{O}(\Omega)$ locally uniformly. Let $U \subset \Omega$ be a bounded open set with $\overline{U} \subset \Omega$, and assume that f has no zeroes on ∂U. Then there exists a number N possibly depending on U, such that*

$$\sum_{c\in U} \mathrm{Ord}(f_n,c) = \sum_{c\in U} \mathrm{Ord}(f,c), \qquad \textit{for all} \quad n \geq N.$$

Proof. Let us consider first the case U is a disk, and set $\varepsilon = \min\limits_{\partial U} |f| > 0$. Choose N so large that $|f_n - f| < \varepsilon$ for all $n > N$. Then Rouché's theorem guarantees that f_n and f have the same number of zeroes (counting multiplicity) in U. When U is a general bounded open set, since f has finitely many zeroes in U, we cover those zeroes by finitely many disjoint open disks and reduce the proof to the case of a disk. $\square$

By applying this theorem to sequences of functions of the form $z \mapsto f_n(z) - w$, we can say various things about $\{f_n\}$ and its limit, as the following theorem shows.

Theorem 19 (Hurwitz injection theorem)**.** *Let $\{f_n\} \subset \mathcal{O}(\Omega)$ be a sequence that converges locally uniformly to a nonconstant $f \in \mathcal{O}(\Omega)$. Then we have*

(a) If all f_n are zero-free, then f is zero-free.
(b) If all f_n are injective, then f is injective.
(c) If there is a set $U \subset \mathbb{C}$ such that $f_n(\Omega) \subseteq U$ for all n, then $f(\Omega) \subseteq U$.

Proof. Part (a) is the special case of (c) with $U = \mathbb{C}^\times$. To prove (c), let $w \in \mathbb{C}\setminus U$ and consider the functions $g_n(z) = f_n(z) - w$, which are nowhere vanishing on Ω. Now if $g(z) = f(z) - w$ had a zero in Ω, then Hurwitz's theorem would imply that almost all g_n have zeroes in Ω. In other words, f cannot take any value from $\mathbb{C} \setminus U$.

As for (b), let $a \in \Omega$ arbitrary and consider the functions $g_n(z) = f_n(z) - f_n(a)$. The injectivity of f_n means that g_n are nowhere vanishing in $\Omega \setminus \{a\}$. Hence $g(z) = f(z) - f(a)$ must be nowhere vanishing in $\Omega \setminus \{a\}$, or in other words, f takes the value $f(a)$ only at a. $\square$

We record here a generalization of the argument principle that gives some information about locations of zeroes and poles. For instance, taking $g(z) = z^n$ in the following theorem one can extract (generalized) moments of the zeroes and poles.

Theorem 20 (Generalized argument principle)**.** *Let $f \in \mathcal{M}(\Omega)$ be nonzero, and let $g \in \mathcal{O}(\Omega)$. Let γ be a null-homologous cycle in Ω. Suppose also that $\mathrm{Ord}(f,z) = 0$ for all $z \in |\gamma|$. Then we have*

$$2\pi i \sum_{c\in \mathrm{Int}\gamma} g(c)\mathrm{Ind}(\gamma,c)\mathrm{Ord}(f,c) = \langle gf'/f, \gamma\rangle,$$

where there are only finitely many nonzero summands in the sum on the left hand side.

Proof. This is a direct computation which gives $\mathrm{Res}(gf'/f,c) = g(c)\mathrm{Ord}(f,c)$. $\square$

5. Riemann surfaces, the Riemann sphere and meromorphic functions

In this section, we extend the domain of complex analysis from planar sets to more general two dimensional manifolds.

We adopt the definition of manifold that an n-dimensional topological *manifold* is a Hausdorff, second countable topological space that is locally homeomorphic to $\mathbb{R}^n$. Recall that a topological space being *Hausdorff* means that any two distinct points have disjoint open neighbourhoods, and *second countable* means that there is a countable collection of open sets such that any open set can be written as a union of sets from the collection.

Definition 21. An abstract, topological surface, or simply a *surface*, is a 2-dimensional connected topological manifold.

Definition 22. A *complex structure* on a surface M is a collection $\mathfrak{C} = \{(U_\alpha, \varphi_\alpha)\}$ of pairs $(U_\alpha, \varphi_\alpha)$ where $U_\alpha \subseteq M$ and $\varphi : U_\alpha \to \mathbb{C}$, satisfying

(1) Each U_α is connected open, and $M = \cup_\alpha U_\alpha$;
(2) Each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{C}$ is a homeomorphism;
(3) Each $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is holomorphic.

Two complex structures $\mathfrak{C}_1$ and $\mathfrak{C}_2$ on M are considered *equivalent* if their union $\mathfrak{C}_1 \cup \mathfrak{C}_2$ is again a complex structure on M.

Definition 23. A *Riemann surface* is a surface with a complex structure. Two Riemann surfaces with equivalent complex structures are considered to be identical.

Connected open sets of $\mathbb{C}$ are trivial examples of Riemann surfaces. For perhaps the simplest nontrivial example, the unit sphere $S^2 \subset \mathbb{R}^3$ can be given a complex structure as follows. Identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$, and call $n = (0, 1)$ the north pole, and $s = (0, -1)$ the south pole. Define the stereographic projections $\varphi_n : S^2 \setminus \{n\} \to \mathbb{C}$ and $\varphi_s : S^2 \setminus \{s\} \to \mathbb{C}$ by

$$\varphi_n(\zeta, \lambda) = \frac{\zeta}{1+\lambda}, \quad (\zeta, \lambda) \in S^2 \setminus \{n\} \qquad \text{and} \qquad \varphi_s(\zeta, \lambda) = \frac{\bar{\zeta}}{1-\lambda}, \quad (\zeta, \lambda) \in S^2 \setminus \{s\}.$$

It is obvious that φ_n and φ_s are homeomorphisms, and that $S^2 \setminus \{n\}$ and $S^2 \setminus \{s\}$ constitute an open cover of S^2. With $z = \varphi_n(\zeta, \lambda)$ and $w = \varphi_s(\zeta, \lambda)$, where z and w are both defined, we have $zw = \frac{|\zeta|^2}{1-\lambda^2} = 1$ since $|\zeta|^2 + \lambda^2 = 1$ on S^2. This means that the transition maps

$$\varphi_s(\varphi_n^{-1}(z)) = 1/z, \quad z \in \mathbb{C}^\times, \qquad \text{and} \qquad \varphi_n(\varphi_s^{-1}(w)) = 1/w, \quad w \in \mathbb{C}^\times,$$

are holomorphic on their respective domains. The Riemann surface thus constructed is called the *Riemann sphere*. Via the map φ_n, the complex plane $\mathbb{C}$ is identified with the Riemann sphere S^2 minus the north pole, and so S^2 can also be thought of as the *extended complex plane* $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, with the complex structure around ∞ defined by that of S^2 around the north pole. The arithmetic operations on $\mathbb{C}$ can be partially extended to $\hat{\mathbb{C}}$, and the role of $\infty \in \hat{\mathbb{C}}$ is somewhat special with respect to those extended operations. Note that however, when $\hat{\mathbb{C}}$ is considered as a Riemann surface, $\infty \in \hat{\mathbb{C}}$ is no more or less a point than any other point in $\hat{\mathbb{C}}$ (The same can be said about the point 0 in $\mathbb{C}$). Indeed, one does not need all of the structural properties of $\mathbb{C}$ in order to define the notion of holomorphy; all one needs is a complex structure. In other words, Riemann surfaces are the natural habitats of holomorphic functions.

Definition 24. Let M and S be Riemann surfaces, with complex structures $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$, respectively. Then $f : M \to S$ is called *holomorphic* if for all α and β, $\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\beta(V_\beta)$ is holomorphic wherever it is defined. The set of holomorphic functions from M to S is denoted by $\mathcal{O}(M, S)$.

In this definition, if $S = \mathbb{C}$ then we simply talk about holomorphic functions on M and write $\mathcal{O}(M) = \mathcal{O}(M, \mathbb{C})$. Almost all of the theorems on domains of $\mathbb{C}$ can be extended to this setting, with some new twists (or rather, insights) including the fact that results on contour integration must now be formulated in terms of holomorphic 1-forms, because on a Riemann surface there is no canonical 1-form that would play the role $\mathrm{d}z$ plays on $\mathbb{C}$. Even when the target surface S is a general Riemann surface, still great many of the theorems carry over, although the fact that one cannot do arithmetic operations on functions preclude certain results (for example, series of functions does not make sense). The case $S = \hat{\mathbb{C}}$ is exceptional in that it is even better than the case $S = \mathbb{C}$ in some sense (see below).

With $D_r \subset \mathbb{C}$ a bounded disk, let $f \in \mathcal{O}(\mathbb{C} \setminus \overline{D})$. If we consider $\mathbb{C} \setminus \overline{D}$ as a subset of $\hat{\mathbb{C}}$, then $\mathbb{C} \setminus \overline{D}$ is a punctured neighbourhood of $\infty \in \hat{\mathbb{C}}$, so the point ∞ is to be considered as an isolated singularity of f. To study this singularity, let us expand f into its Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

This expansion is not immediately adequate for examining the point ∞ because the z-coordinate chart does not contain ∞. So we perform the coordinate change $w = 1/z$ and work with the w-coordinates. In these coordinates $\infty \in \hat{\mathbb{C}}$ is $w = 0$, and the above-displayed Laurent expansion becomes

$$f(w) = \sum_{n=-\infty}^{\infty} a_n w^{-n} = \sum_{n=-\infty}^{\infty} a_{-n} w^n.$$

Now it is clear how the singularity at ∞ should be classified given the coefficients a_n of the Laurent series around $0 \in \hat{\mathbb{C}}$. The point ∞ is

- a removable singularity if $a_n = 0$ for all $n > 0$;
- a pole of order N if $a_N \neq 0$ and $a_n = 0$ for all $n > N$;
- an essential singularity if $a_n \neq 0$ for infinitely many $n > 0$.

For example, polynomials have poles at ∞, and the only holomorphic functions sending $\hat{\mathbb{C}}$ to $\mathbb{C}$ are constants. It also follows that if we allow only poles (and no essential singularities) in $f : \hat{\mathbb{C}} \to \mathbb{C}$, then f must be a rational function, i.e., f must be a quotient of two polynomials.

The set of holomorphic functions $\mathcal{O}(M)$ on a Riemann surface M is a ring with units (i.e., invertible elements) given by nowhere vanishing functions. What if we try to (multiplicatively) invert holomorphic functions with zeroes? The question is not that hopeless if we remember that zeroes of a nonzero holomorphic function form a discrete set, and from the characterization of poles theorem (Theorem 4 on page 3) that zeroes become poles and poles become zeroes upon invertion. This motivates the following definition, whose special case we have already encountered in the preceding section.

Definition 25. Let M be a Riemann surface, and let $K \subset M$ be a discrete set. Then $f : M \setminus K \to \mathbb{C}$ is called *meromorphic* on M if $f \in \mathcal{O}(M \setminus K)$ and f has a pole at each point of K in local coordinates. The set of meromorphic functions on M is denoted by $\mathcal{M}(M)$.

From Theorem 4 it follows now that any nonzero element of $\mathcal{M}(M)$ is invertible, i.e., $\mathcal{M}(M)$ is a field. Also from the same theorem, we infer that if $f \in \mathcal{M}(M)$ then near a pole $p \in M$ of f, $1/f$ is holomorphic and has a zero at p. Hence, meromorphic functions on M are indeed holomorphic functions from M to $\hat{\mathbb{C}}$. Since a nonconstant holomorphic function takes any fixed value (in particular the value ∞) on at most a discrete set, one concludes that holomorphic functions from M to $\hat{\mathbb{C}}$ are meromorphic on M, so that $\mathcal{M}(M) = \mathcal{O}(M, \hat{\mathbb{C}})$.