MATH 566 LECTURE NOTES 3: THE FOUNDATIONS OF CAUCHY'S FUNCTION THEORY

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1. Introduction

In this set of notes, we continue to explore the fundamental theorems of complex analysis. Among others, we will prove and discuss the following result.

Theorem 1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in C(\Omega)$. Then the followings are equivalent.*
(a) f is holomorphic on Ω, i.e., $f \in \mathcal{O}(\Omega)$.
(b) For all closed triangles $T \subset \Omega$, the integral of f over the boundary of T is zero.
(c) f is locally integrable in Ω, i.e., for each $z \in \Omega$ there exist an open neighbourhood U of z and a function $F \in \mathcal{O}(U)$ such that $F' = f$ on U.
(d) For all closed disks $\overline{D} \subset \Omega$ and for $a \in D$, it holds that $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{f(z)\mathrm{d}z}{z-a}$.
(e) f is analytic on Ω, i.e., $f \in C^\omega(\Omega)$.

The implication (e) $\Rightarrow$ (a) follows from the termwise differentiation theorem that has been proven in the previous set of notes on analytic functions. The implication (a) $\Rightarrow$ (b) is Goursat's theorem in §3, and (b) $\Rightarrow$ (c) is Theorem 8 in §4. Then (a) $\Rightarrow$ (d) is the argument leading to Cauchy's integral formula in §5 and §8, where we also use the implication (a) $\Rightarrow$ (c), and (d) $\Rightarrow$ (e) is the Cauchy-Taylor theorem in §9. Using all of these, (c) $\Rightarrow$ (a) is proven as Morera's theorem in §9. Theorems having a conclusion of type (b) with triangles replaced by more general curves are typically called *Cauchy theorems*, and we will prove several versions of Cauchy theorems in §5 and §6. Statements of type (d) are called *Cauchy integral formulæ*; we shall prove a couple of those in §8, after presenting the *residue theorem* in §7, which is a generalization of both Cauchy theorems and Cauchy integral formulæ. Let us start by fixing some terminology regarding curves, and clarifying the notion of integrating a complex-valued function over curves.

2. Contour integration

Let $\Omega \subseteq \mathbb{C}$ be an open set. A (topological) *curve* in Ω is simply a continuous map $\gamma : [a,b] \to \Omega$, and it is called a *closed curve* or a *loop* if $\gamma(a) = \gamma(b)$. Loops in Ω can also be defined as continuous maps $\gamma : S^1 \to \Omega$. The terms path, contour and arc are also used for a curve, sometimes with slight differences in meaning. We will not make any distinction between any of these terms. Non-self-intersecting curves are called *simple*, and simple closed curves are called *Jordan curves*. The *length* of a curve $\gamma \in C([a,b],\Omega)$, the latter notation meaning that the space of continuous functions on $[a,b]$ taking values from Ω, is defined as

$$\|\gamma\| = \sup_P \sum_{k=1}^n |\gamma(t_i) - \gamma(t_{i-1})|,$$

where the supremum is taken over all possible partitions $P = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ with $a = t_0 < t_1 < \ldots < t_n = b$. Finite length curves are said to be *rectifiable*. Integration over

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rectifiable curves is possible via the *Riemann-Stieltjes integrals*, but we will consider here a smoother class of curves for which the theory of Riemann integrals suffices.

If $\phi : [c, d] \to [a, b]$ is a monotone increasing surjective function, then we say the curve $\gamma \circ \phi : [c, d] \to \Omega$ is equivalent to the original $\gamma : [a, b] \to \Omega$, and call the equivalence classes of curves under this equivalence relation *oriented curves*. Intuitively, given the *image* $|\gamma| = \gamma([a, b])$ of the curve γ, an oriented curve can be recovered upon identifying the initial and terminal points, and specifying how to traverse at self-intersection points. By abuse of language we call the particular representation $\gamma : [a, b] \to \Omega$ of the underlying oriented curve also an oriented curve. Note that one can take the interval $[a, b]$ to be, say, $[0, 1]$ at one's convenience. Now, the *inverse* or the *opposite* of γ is defined by reversing the orientation: $\gamma^{-1}(t) = \gamma(b + a - t)$ for $t \in [a, b]$. If $\gamma : [0, 1] \to \Omega$ and $\sigma : [1, 2] \to \Omega$ are two curves with $\gamma(1) = \sigma(1)$, then their *product* or *concatenation* $\gamma\sigma : [0, 2] \to \Omega$ is defined as $\gamma\sigma(t) = \gamma(t)$ for $t \in [0, 1]$ and $\gamma\sigma(t) = \sigma(t)$ for $t \in [1, 2]$. When the order of the operations are not important, the above operations on curves can suggestively be written in the additive notation as $-\gamma \equiv \gamma^{-1}$ and $\gamma + \sigma \equiv \gamma\sigma$.

The curve $\gamma : [a, b] \to \Omega$ is called *differentiable* if $\gamma \in C^1([a, b])$ (with $\gamma'(a) = \gamma'(b)$ for loops) and $\gamma'(t) \neq 0$ for $t \in [a, b]$, where the derivatives at a and b are to be understood in the one-sided sense. The curve γ is called *piecewise differentiable* in Ω and written $\gamma \in C^1_{\text{pw}}([a, b], \Omega)$ if γ is the concatenation of finitely many differentiable curves. We assume that differentiable and piecewise differentiable curves are oriented, which amounts to saying, e.g., for the case of piecewise differentiable curves that we allow piecewise differentiable monotone reparameterizations of curves.

Let $\alpha = p\mathrm{d}x + q\mathrm{d}y$ be a differential 1-form on Ω having values in some finite dimensional vector space V, i.e., the components p and q are V-valued functions on Ω. The integral of α over a differentiable curve $\gamma : [a, b] \to \Omega$ is an element of V, defined by

$$\langle \alpha, \gamma \rangle = \int_\gamma \alpha = \int_a^b \left[p(\gamma(t))\gamma_x'(t) + q(\gamma(t))\gamma_y'(t) \right] \mathrm{d}t,$$

where γ_x and γ_y are the x- and y-components of γ. One can verify that this definition is invariant under differentiable monotone reparameterizations of γ. We will use the notation $\langle \alpha, \gamma \rangle$ for the integral of α over γ wherever it is not too awkward, which is more in line with the duality between the differential form and the domain of integration than the conventional notation. For piecewise differentiable curves the integral is defined via "linearity":

$$\langle \alpha, \gamma_1 + \ldots + \gamma_n \rangle = \langle \alpha, \gamma_1 \rangle + \ldots + \langle \alpha, \gamma_n \rangle.$$

If φ is a V-valued 0-form on Ω, i.e., if $\varphi : \Omega \to V$, its exterior derivative is defined to be the 1-form $\mathrm{d}\varphi = \partial_x \varphi \mathrm{d}x + \partial_y \varphi \mathrm{d}y$. Note that we have implicitly assumed the existence of the derivatives of φ. We recall below a version of the fundamental theorem of calculus.

Theorem 2 (Fundamental theorem of calculus). *Let $\varphi : \Omega \to V$ be a continuously differentiable function, and let $\gamma \in C^1_{\text{pw}}([a, b], \Omega)$ be a piecewise differentiable oriented curve in Ω. Then we have*

$$\int_\gamma \mathrm{d}\varphi = \varphi(\gamma(b)) - \varphi(\gamma(a)).$$

If we consider the *boundary* $\partial\gamma$ of γ as the formal linear combination of the initial point $\gamma(a)$ with weight -1 and the terminal point $\gamma(b)$ with weight $+1$, i.e., $\partial\gamma = \gamma(b) - \gamma(a)$, and moreover if we define the evaluation (or "integral") of the 0-form φ on formal linear combinations $k_1 z_1 + \ldots + k_n z_n$ of points $z_1, \ldots, z_n \in \Omega$ with weights $k_1, \ldots, k_n \in \mathbb{Z}$, again by "linearity" as $\langle \varphi, k_1 z_1 + \ldots + k_n z_n \rangle = k_1 \varphi(z_1) + \ldots + k_n \varphi(z_n)$, then the conclusion of the above theorem can be written compactly as $\langle \mathrm{d}\varphi, \gamma \rangle = \langle \varphi, \partial\gamma \rangle$. Finite formal linear combinations $k_1 z_1 + \ldots + k_n z_n$ of points $z_1, \ldots, z_n \in \Omega$ with weights $k_1, \ldots, k_n \in \mathbb{Z}$ are called 0-*chains* in Ω. The collection of all 0-chains in Ω forms a free abelian group $C_0(\Omega)$, called the *0-th chain group* of Ω. Similarly, the notion of curves can be extended to *1-chains*, which are finite formal linear combinations $k_1\gamma_1 + \ldots + k_n\gamma_n$ of oriented curves $\gamma_1, \ldots, \gamma_n \in C([0,1],\Omega)$ with weights $k_1, \ldots, k_n \in \mathbb{Z}$. The set of 1-chains forms a free abelian group $C_1(\Omega)$, called the *1-st chain group* of Ω. The boundary operator ∂ can be extended to 1-chains by linearity

$$\partial(k_1\gamma_1 + \ldots + k_n\gamma_n) = k_1\partial\gamma_1 + \ldots + k_n\partial\gamma_n,$$

so that $\partial : C_1(\Omega) \to C_0(\Omega)$ is now a group homomorphism. Also the integral over piecewise differentiable 1-chains (i.e., 1-chains that can be written as linear combinations of piecewise differentiable curves) can be defined via linearity as $\langle \alpha, k_1\gamma_1 + \ldots + k_n\gamma_n\rangle = k_1\langle\alpha,\gamma_1\rangle + \ldots + k_n\langle\alpha,\gamma_n\rangle$. With the above concepts at hand, the fundamental theorem of calculus immediately generalizes to 1-chains: If $\varphi : \Omega \to V$ is continuously differentiable, then there holds that $\langle \mathrm{d}\varphi, \gamma\rangle = \langle\varphi, \partial\gamma\rangle$ for piecewise differentiable $\gamma \in C_1(\Omega)$.

Let us now consider the special case $V = \mathbb{C}$. In this case, making use of the algebraic structure of $\mathbb{C}$, it is convenient to express everything in terms of the elementary differential forms $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - i\mathrm{d}y$. For instance, we have

$$\alpha = p\mathrm{d}x + q\mathrm{d}y = p\frac{\mathrm{d}z + \mathrm{d}\bar{z}}{2} + q\frac{\mathrm{d}z - \mathrm{d}\bar{z}}{2i} = \frac{p - qi}{2}\mathrm{d}z + \frac{p + qi}{2}\mathrm{d}\bar{z} =: \lambda\mathrm{d}z + \mu\mathrm{d}\bar{z},$$

i.e., any $\mathbb{C}$-valued 1-form can be written as $\lambda\mathrm{d}z + \mu\mathrm{d}\bar{z}$ with $\mathbb{C}$-valued functions λ and μ. Conversely, any pair of functions λ and μ defines a 1-form. Among all $\mathbb{C}$-valued 1-forms, we are mainly interested in *holomorphic 1-forms*, which are of the form $\alpha = \lambda\mathrm{d}z$ with a holomorphic function λ. Furthermore, for a holomorphic (or more generally a complex-valued) function λ, we define the *integral* of λ over a piecewise differentiable curve (or more generally a piecewise differentiable 1-chain) γ to be the integral $\langle\lambda\mathrm{d}z, \gamma\rangle$, and if there is no risk of confusion we write it simply as $\langle\lambda,\gamma\rangle$.

Let φ be a $\mathbb{C}$-valued 0-form (i.e., a garden-variety complex function). We have

$$\mathrm{d}\varphi = \partial_x\varphi\mathrm{d}x + \partial_y\varphi\mathrm{d}y = \frac{\partial_x\varphi - i\partial_y\varphi}{2}\mathrm{d}z + \frac{\partial_x\varphi + i\partial_y\varphi}{2}\mathrm{d}\bar{z} = \partial_z\varphi\,\mathrm{d}z + \partial_{\bar{z}}\varphi\,\mathrm{d}\bar{z} =: \partial\varphi + \bar{\partial}\varphi,$$

and if φ is holomorphic, then $\partial_{\bar{z}}\varphi = 0$, so that $\mathrm{d}\varphi = \partial_z\varphi\,\mathrm{d}z = \partial\varphi$. By applying the fundamental theorem of calculus to such functions, we get the following useful theorem. In particular, this theorem will be used to show that if f is an *integrable* holomorphic function, i.e., if there exists a holomorphic function F such that $\partial_z F = f$ throughout Ω, then the integral of f over any closed curve in Ω is zero.

Theorem 3 (FTC for holomorphic functions)**.** *Let $\varphi \in \mathcal{O}(\Omega)$ be a holomorphic function, and suppose that $\partial_z\varphi$ is continuous in Ω. Then for any piecewise differentiable $\gamma \in C_1(\Omega)$ we have*

$$\langle\partial_z\varphi, \gamma\rangle = \langle\varphi, \partial\gamma\rangle.$$

Proof. We only have to show that $\partial_x\varphi$ and $\partial_y\varphi$ are continuous in Ω. But this is true since by definition $\partial_{\bar{z}}\varphi \equiv 0$ and by hypothesis $\partial_z\varphi$ is continuous throughout Ω. □

Remark 4. The continuity hypothesis on $\partial_z\varphi$ is in fact superfluous, since it will turn out that holomorphic functions are analytic, so in particular they are infinitely often differentiable. However, the above form (with the continuity hypothesis) will be used to prove that fact.

3. Goursat's theorem

For a set $U \subset \mathbb{C}$ that is not open, the notation $f \in \mathcal{O}(U)$ means that f is holomorphic in an open neighbourhood of U. Let us denote by $[z_1z_2]$ the (oriented) line segment with the initial point $z_1 \in \mathbb{C}$ and the terminal point $z_2 \in \mathbb{C}$, and by $[z_1z_2z_3]$ the (possibly degenerate) triangle with vertices $z_k \in \mathbb{C}$. The *boundary* $\partial\tau$ of $\tau = [z_1z_2z_3]$ is defined to be the loop $[z_1z_2] + [z_2z_3] + [z_3z_1]$. Note that the orientation of the boundary depends on the order of the vertices in $[z_1z_2z_3]$, so for example, $\partial[z_1z_2z_3] = \partial[z_2z_3z_1] = -\partial[z_2z_1z_3]$. If the interior of τ is on the left of $\partial\tau$, then we say that τ is *positively oriented*, otherwise – negatively oriented. The following is called *Goursat's theorem*.

Theorem 5. *Let $\tau \subset \mathbb{C}$ be a triangle, and let $f \in \mathcal{O}(\overline{\tau})$. Then $\langle f, \partial\tau\rangle = 0$.*

Proof. Let us subdivide τ into 4 congruent triangles $\tau_1, \tau_2, \tau_3, \tau_4$ by connecting the midpoints of the edges of τ. All lengths of the smaller triangles are measured as half the corresponding length of the original triangle τ. Moreover we have

$$\langle f, \partial\tau\rangle = \sum_{1\leq j\leq 4} \langle f, \partial\tau_j\rangle.$$

Let τ_m be a triangle among the 4 triangles that gives the largest contribution to the sum, and call it $\tau^{(1)}$, that is, τ_m (with some m between 1 and 4) satisfies $|\langle f, \partial\tau_m\rangle| \geq |\langle f, \partial\tau_j\rangle|$ for any $1 \leq j \leq 4$. Then we have

$$|\langle f, \partial\tau\rangle| \leq 4|\langle f, \partial\tau^{(1)}\rangle|.$$

Now subdividing $\tau^{(1)}$ into 4 still smaller triangles, and repeating this procedure, we get

$$|\langle f, \partial\tau\rangle| \leq 4^n|\langle f, \partial\tau^{(n)}\rangle|, \tag{1}$$

with any length of $\tau^{(n)}$ being 2^{-n} part of the corresponding length of τ. In particular, if c_n is the barycenter of $\tau^{(n)}$, then the sequence $\{c_n\}$ is Cauchy, so $c_n \to c \in \overline{\tau}$. Since f is holomorphic in a neighbourhood of $\overline{\tau}$, by definition we have

$$f(z) = f(c) + \lambda(z - c) + o(2^{-n}), \qquad z \in \partial\tau^{(n)},$$

with some constant $\lambda \in \mathbb{C}$. We calculate the integral of f over the boundary of $\tau^{(n)}$ to be

$$\langle f, \partial\tau^{(n)}\rangle = (f(c) - \lambda c)\int_{\partial\tau^{(n)}} \mathrm{d}z + \lambda\int_{\partial\tau^{(n)}} z\mathrm{d}z + o(2^{-n}\|\partial\tau^{(n)}\|) = o(4^{-n}),$$

where the displayed integrals vanish because of the integrability of 1 an z, and we have taken into account that the perimeter $\|\partial\tau^{(n)}\| = O(2^{-n})$. Substituting this into (1) establishes the proof. $\square$

It is possible to slightly relax the hypothesis of Goursat's theorem, so that only holomorphy in the interior and continuity up to the boundary are assumed. The argument is a continuity argument that can be used to strengthen many of the theorems that follow.

Corollary 6. *Let $\tau \subset \mathbb{C}$ be an open triangle, and let $f \in \mathcal{O}(\tau) \cap C(\overline{\tau})$. Then $\langle f, \partial\tau\rangle = 0$.*

Proof. Let a, b, c be the vertices of τ, and let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of points in τ such that $a_n \to a$, $b_n \to b$, and $c_n \to c$ as $n \to \infty$. As the closure of the triangle $\tau_n = [a_nb_nc_n]$ is entirely in τ, Goursat's theorem applies to τ_n, meaning that $\langle f, \partial\tau_n\rangle = 0$. By uniform continuity, $\langle f, \partial\tau_n\rangle$ tends to $\langle f, \partial\tau\rangle$, hence $\langle f, \partial\tau\rangle = 0$. $\square$

4. Local integrability

In what follows, by default Ω will always denote an open subset of $\mathbb{C}$.

Definition 7. A continuous function $f \in C(\Omega)$ is called *integrable* on Ω if there is $F \in \mathcal{O}(\Omega)$ such that $\partial_z F = f$ on Ω. It is called *locally integrable* on Ω if for any $z \in \Omega$ there exists a neighbourhood U of z such that f is integrable on U.

In combination with Goursat's theorem, the theorem below implies that holomorphic functions are locally integrable.

Theorem 8. *Let $f \in C(\Omega)$ and suppose that $\langle f, \partial\tau\rangle = 0$ for any closed triangle $\tau \subset \Omega$. Then f is integrable on any open disk $D \subseteq \Omega$.*

Proof. Let $c \in D$ be the centre of D, and define $F(z) = \langle f, [cz] \rangle$ for $z \in D$. We would like to show that $F' = f$ on D, or equivalently that

$$F(w) = F(z) + f(z)(w - z) + o(|w - z|).$$

From the definition of F we have $F(w) - F(z) = \langle f, [zw] \rangle$, and taking into account that $w - z = \langle 1, [zw] \rangle$, we infer

$$F(w) - F(z) - f(z)(w - z) = \langle f, [zw] \rangle - f(z)\langle 1, [zw] \rangle.$$

Now $f = f(z) + o_{|w-z|}(1)$ on $[zw]$, so the right hand side is of order $o(|w - z|)$. □

In the subsequent sections, by a sequence of several theorems, we will prove that locally integrable functions are analytic, therefore also holomorphic, cf. Theorem 31 on page 11. Hence local integrability is equivalent to holomorphy.

As a simple application of the theorem, we get Cauchy's theorem for disks.

Corollary 9. *Let $f \in \mathcal{O}(\Omega)$ and let $D \subseteq \Omega$ be an open disk. Then $\langle f, \gamma \rangle = 0$ for any piecewise differentiable loop $\gamma \in C^1_{\mathrm{pw}}(S^1, D)$ lying in D.*

Proof. By the preceding theorem (in combination with Goursat's theorem) there is $F \in \mathcal{O}(\Omega)$ such that $F' = f$ on D. Then the fundamental theorem of calculus for holomorphic functions (Theorem 3 on page 3) states that the integral of f over any piecewise differentiable closed curve must be zero. □

We can slightly extend the argument in the proof of Theorem 8 to get a criterion on (global) integrability.

Theorem 10. *A continuous function $f \in C(\Omega)$ is integrable on Ω if and only if $\langle f, \gamma \rangle = 0$ for any $\gamma \in C^1_{\mathrm{pw}}(S^1, \Omega)$.*

Proof. One direction is immediate from the fundamental theorem of calculus. For the other direction, assume that Ω is connected (otherwise we work in connected components of Ω one by one). Let $c \in \Omega$, and for $z \in \Omega$ define $F(z) = \langle f, \gamma \rangle$ with γ a piecewise differentiable curve connecting[1] c and z. The value $F(z)$ does not depend on the particular curve γ, since if σ is another curve connecting c and z, then $\gamma - \sigma$ is a piecewise differentiable loop in Ω, so that $\langle f, \gamma \rangle = \langle f, \sigma \rangle$ by hypothesis. Now noting that $F(w) - F(z) = \langle f, [zw] \rangle$, the proof proceeds in exactly the same way as in the proof of Theorem 8. □

5. Cauchy's theorem for homotopic loops

Definition 11. Given a set $F \subset [0, 1]$, curves $\gamma_0, \gamma_1 \in C([0, 1], \Omega)$ are called *homotopic relative to* F, and written $\gamma_0 \eqsim_F \gamma_1$, if there exists a continuous map $\Gamma : [0, 1] \times [0, 1] \to \Omega$ such that $\Gamma(t, 0) = \gamma_0(t)$ and $\Gamma(t, 1) = \gamma_1(t)$ for $t \in [0, 1]$, and $\Gamma(t, s) = \gamma_0(t)$ for all $t \in F$ and $s \in [0, 1]$.

Note that $\gamma_0 \eqsim_F \gamma_1$ implies in particular that $\gamma_0(t) = \gamma_1(t)$ for all $t \in F$. Homotopy relative to F is an equivalence relation in the space of curves that are "fixed" at F, and so this space is partitioned into (relative) *homotopy classes*. Let us first show that when F is finite, every such a class contains a polygonal (i.e., piecewise linear) representative.

Lemma 12. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $\gamma \in C([0, 1], \Omega)$ be a curve in Ω. Then for any finite set $F \subset [0, 1]$, there is a polygonal path σ in Ω such that $\sigma \eqsim_F \gamma$. Moreover, given an open cover $\{U_j\}$ of $|\gamma|$, the polygonal path σ can be chosen to be also covered by $\{U_j\}$.*

[1]Any two points in a connected open planar set can be connected by a piecewise linear curve.

Proof. We prove the lemma for the case $F = \{0, 1\}$. Since $[0, 1]$ is compact, γ is uniformly continuous, and the image $|\gamma| = \{\gamma(t) : t \in [0, 1]\}$ is compact in Ω. Fix $\varepsilon > 0$ such that $\varepsilon < \mathrm{dist}(|\gamma|, \mathbb{C} \setminus \Omega)$, and that $\cup_t D_\varepsilon(\gamma(t)) \subset \cup_j U_j$. Moreover, for an integer n that will be chosen shortly, let $z_j = \gamma(j/n)$ for $j = 0, \ldots, n$. Let $\sigma_j = [z_j z_{j+1}]$ be the line segment joining z_j and z_{j+1}, and let $\gamma_j = \gamma|_{[j/n,(j+1)/n]}$. Then by uniform continuity, for sufficiently large n one has $\sigma_j \subset D_\varepsilon(z_j)$ and $\gamma_j \subset D_\varepsilon(z_j)$ for all j. Without loss of generality assuming that σ_j is parameterized by the interval $[j/n, (j + 1)/n]$, the map $\Gamma_j(t, s) = (1 - s)\sigma_j(t) + s\gamma(t)$ defines a homotopy between σ_j and γ_j relative to their endpoints. This shows that γ is homotopic to the polygonal path $\sigma = \sigma_1 + \ldots + \sigma_n$ relative to their endpoints. □

Corollary 13. *Let f be locally integrable in Ω, and let $\gamma \in C^1_{\mathrm{pw}}([0, 1], \Omega)$ be a piecewise differentiable curve. Then there is a polygonal path $\sigma \eqsim_{\{0,1\}} \gamma$ such that $\langle f, \sigma\rangle = \langle f, \gamma\rangle$.*

Proof. Let $\{D_j\}$ be a finite cover of $|\gamma|$ by open disks such that f is integrable on each D_j. Choose a partition $0 = t_0 < \ldots < t_n = 1$ such that $\gamma([t_j, t_{j+1}]) \subset D_j$ for each j. Then by the preceding lemma there is a polygonal path $\sigma : [0, 1] \to \Omega$ covered by $\{D_j\}$, such that σ meets γ at each of the points $\gamma(t_j)$. If we set $\sigma_j = \sigma|_{[t_j,t_{j+1}]}$ and $\gamma_j = \gamma|_{[t_j,t_{j+1}]}$, then since

$$\langle f, \gamma\rangle - \langle f, \sigma\rangle = \sum_j \langle f, \gamma_j - \sigma_j\rangle,$$

and $\gamma_j - \sigma_j$ is a closed curve entirely contained in D_j, on which f is integrable, by the fundamental theorem of calculus we have $\langle f, \gamma_j - \sigma_j\rangle = 0$. This completes the proof. □

This corollary points to the possibility of defining the integral $\langle f, \gamma\rangle$ of a holomorphic function f when γ is only a continuous curve by replacing γ with a polygonal path $\sigma \eqsim_{\{0,1\}} \gamma$. For this definition to be meaningful it has to hold that $\langle f, \gamma_0\rangle = \langle f, \gamma_1\rangle$ for any two piecewise differentiable curves $\gamma_0 \eqsim_{\{0,1\}} \gamma_1$. We will accomplish this by first proving a more general statement about integration over homotopic loops.

Definition 14. Loops $\gamma_0, \gamma_1 \in C(S^1, \Omega)$ are called (freely) *homotopic* to each other, and written $\gamma_0 \eqsim \gamma_1$, if there exists a continuous map $\Gamma : S^1 \times [0, 1] \to \Omega$ such that $\Gamma(t, 0) = \gamma_0(t)$ and $\Gamma(t, 1) = \gamma_1(t)$ for $t \in S^1$.

Similarly to the relative homotopy case, the space of loops is partitioned into (free) *homotopy classes*. The following theorem shows that at least in the piecewise differentiable case, the integral of a given holomorphic function over a loop depends only on the homotopy class the loop represents. In other words, denoting by $\Pi(\Omega)$ the set of homotopy classes of loops in Ω, any holomorphic function f on Ω induces a well-defined function on $\Pi(\Omega)$ by $\gamma \mapsto \langle f, \gamma\rangle$, where γ is a (piecewise differentiable) representative of the homotopy class $[\gamma] \in \Pi(\Omega)$.

Theorem 15. *For $f \in \mathcal{O}(\Omega)$ and for piecewise differentiable loops $\gamma_0, \gamma_1 \in C^1_{\mathrm{pw}}(S^1, \Omega)$ with $\gamma_0 \eqsim \gamma_1$, we have*

$$\langle f, \gamma_0\rangle = \langle f, \gamma_1\rangle.$$

Proof. Since holomorphic functions are locally integrable by Goursat's theorem and Theorem 8 on page 4, by the preceding corollary it suffices to prove the theorem for polygonal paths γ_0 and γ_1. Moreover, let us parametrize the circle S^1 by the interval $[0, 1]$, so the curves will be maps defined on $[0, 1]$. Let $\Gamma : [0, 1]^2 \to \Omega$ be a homotopy between γ_0 and γ_1. Since $[0, 1]^2$ is compact, Γ is uniformly continuous, and the image $|\Gamma| = \{\Gamma(t, s) : (t, s) \in [0, 1]^2\}$ is compact in Ω. Fix $\varepsilon > 0$ such that $\varepsilon < \mathrm{dist}(|\Gamma|, \mathbb{C} \setminus \Omega)$ and that f is integrable on any disk $D_\varepsilon(z)$ with $z \in |\Gamma|$. For a large integer n, let $z_{j,k} = \Gamma(\frac{j}{n}, \frac{k}{n})$ for $j = 0, \ldots, n$, and $k = 0, \ldots, n$. Let $Q_{j,k}$ be the oriented polygonal loop with the vertices $z_{j,k}$, $z_{j+1,k}$, $z_{j+1,k+1}$, and $z_{j,k+1}$ in the same order. We choose n to be so large that $Q_{j,k} \subset D_\varepsilon(z_{j,k})$ for all j and k. Then defining the polygonal loop σ_k to be the one with the vertices $z_{0,k}, \ldots, z_{n-1,k}$ for $k = 0, \ldots, n$, we claim that $\langle f, \sigma_0\rangle = \langle f, \sigma_n\rangle$. Note that

$$\langle f, \sigma_0\rangle - \langle f, \sigma_n\rangle = \sum_{j,k} \langle f, Q_{j,k}\rangle,$$

where the contribution from any edge of $Q_{j,k}$ that does not coincide with an edge of either σ_0 or σ_n is canceled due to the opposite orientations that a common edge inherits from neighbouring polygons. Moreover, each integral $\langle f, Q_{j,k}\rangle$ is zero because f is integrable on $D_\varepsilon(z_{j,k}) \subset \Omega$ and $Q_{j,k}$ is a polygonal loop in $D_\varepsilon(z_{j,k})$. The claim is proven.

It remains to show that $\langle f, \gamma_0\rangle = \langle f, \sigma_0\rangle$ and $\langle f, \gamma_1\rangle = \langle f, \sigma_n\rangle$. This can be done by adjusting the grid on $[0, 1]$ so that the vertices of γ_0 and γ_1 correspond to grid points, and so making sure that $\sigma_0 = \gamma_0$ and $\sigma_n = \gamma_1$. $\square$

If a loop γ is homotopic to a constant path, i.e., $\gamma \approx \delta$ with $\delta : [a, b] \to \Omega$ such that $\delta \equiv z$ for some $z \in \Omega$, then γ is said to be *topologically trivial* or *homotopic to zero*, and this fact is written as $\gamma \approx 0$.

Corollary 16. *If $\gamma \in C^1_{\mathrm{pw}}(S^1, \Omega)$ is topologically trivial, then $\langle f, \gamma\rangle = 0$ for any $f \in \mathcal{O}(\Omega)$.*

Finally, we prove the result we alluded to earlier that the integral of holomorphic functions can be defined over a continuous curve γ by replacing γ with a polygonal path $\sigma \approx_{\{0,1\}} \gamma$.

Corollary 17. *For $f \in \mathcal{O}(\Omega)$ and for piecewise differentiable curves $\gamma_0, \gamma_1 \in C^1_{\mathrm{pw}}([0, 1], \Omega)$ with $\gamma_0 \approx_{\{0,1\}} \gamma_1$, we have $\langle f, \gamma_0\rangle = \langle f, \gamma_1\rangle$.*

Proof. One can show that the curve $\gamma_0 - \gamma_1$ is a topologically trivial piecewise differentiable loop, by constructing a homotopy that, e.g., first follows the homotopy between γ_0 and γ_1 relative to the endpoints to collapse γ_0 onto γ_1, and then contracts γ_1 to a point. $\square$

A somewhat trivial way to ensure that a particular closed curve in Ω is topologically trivial is to simply require that every closed curve in Ω is topologically trivial.

Definition 18. $\Omega \subseteq \mathbb{C}$ is called *simply connected* if it is connected and every closed curve in Ω is topologically trivial.

Example 19. Convex sets are simply connected. More general simply connected sets are star-shaped sets, which are characterized by the property that there is $c \in \Omega$ such that $z \in \Omega$ implies $[zc] \subset \Omega$.

Corollary 20. *If Ω is simply connected then $\langle f, \gamma\rangle = 0$ for $f \in \mathcal{O}(\Omega)$ and $\gamma \in C^1_{\mathrm{pw}}(S^1, \Omega)$.*

6. Cauchy's theorem for homologous cycles

As we have seen in §2, integrals over curves can be generalized to integrals over 1-chains. Cauchy's theorem has a very natural form in the language of chains. In this setting, loops shall be replaced by more general objects called cycles, and homotopy shall be replaced by a weaker notion of homology.

Definition 21. A chain $\gamma \in C_1(\Omega)$ is called a *cycle* if $\partial\gamma = 0$, and the set of all cycles is denoted by $Z_1(\Omega)$. If $\gamma \in Z_1(\Omega)$ can be written as $\gamma = k_1\gamma_1 + \ldots + k_n\gamma_n$ with each γ_j a topologically trivial loop, then γ is called a *null-homologous* or *homologically trivial* cycle, and written $\gamma \sim 0$. Two cycles $\gamma_0, \gamma_1 \in Z_1(\Omega)$ are said to be *homologous* to each other, and written $\gamma_0 \sim \gamma_1$, if they differ on a null-homologous cycle, i.e., if $\gamma_0 - \gamma_1 \sim 0$.

Being the kernel of the group homomorphism $\partial : C_1(\Omega) \to C_0(\Omega)$, the set of cycles $Z_1(\Omega) = \{\gamma \in C_1(\Omega) : \partial\gamma = 0\}$ forms a subgroup of the chain group $C_1(\Omega)$. Homology between cycles is an equivalence relation, so the cycle group $Z_1(\Omega)$ is decomposed into *homology classes*.

We write $[\gamma]$ for the homology class containing the cycle $\gamma \in Z_1(\Omega)$. Since $B_1(\Omega) = \{\gamma \in Z_1(\Omega) : \gamma \sim 0\}$ is a subgroup of $Z_1(\Omega)$, the set of homology classes forms a factor group, called the *first homology group* of Ω, denoted by $H_1(\Omega) = Z_1(\Omega)/B_1(\Omega)$. Moreover, while the groups $Z_1(\Omega)$ and $B_1(\Omega)$ are free abelian essentially by definition, it is a nontrivial topological fact that, e.g., when Ω has finitely many holes (i.e., when $\mathbb{C} \setminus \Omega$ has finitely many connected components), $H_1(\Omega)$ is also free abelian[2].

The following theorem says that the integral of a holomorphic function over a cycle is uniquely determined by the homology class of the cycle.

Theorem 22. *For $f \in \mathcal{O}(\Omega)$, and for piecewise differentiable cycles $\gamma_0, \gamma_1 \in Z_1(\Omega)$ with $\gamma_0 \sim \gamma_1$, we have*

$$\langle f, \gamma_0 \rangle = \langle f, \gamma_1 \rangle.$$

Proof. It suffices to prove that $\langle f, \gamma \rangle = 0$ for $\gamma \sim 0$. By definition $\gamma = k_1\gamma_1 + \ldots + k_n\gamma_n$ with each $\gamma_j \approx 0$, and by Lemma 12 there is a polygonal loop $\sigma_j \approx \gamma_j$ for each j. Then an application of Cauchy's theorem for homotopic loops concludes the proof. □

Considered as cycles, homotopic loops are homologous, but homologous loops are not necessarily homotopic (find an example!). So the preceding theorem is stronger than Cauchy's theorem for homotopic loops. However, it turns out that if the underlying domain is simply connected, then the two theorems are equivalent.

Lemma 23. *Every cycle $\gamma \in Z_1(\Omega)$ can be written as*

$$\gamma = k_1\gamma_1 + \ldots + k_n\gamma_n,$$

where all γ_j are closed oriented curves.

Proof. Since γ is in particular a 1-chain, we can write

$$\gamma = k_1\gamma_1 + \ldots + k_n\gamma_n,$$

with oriented curves γ_j. By replacing γ_j with $-\gamma_j$ if necessary, we can assume that all $k_j > 0$. We apply induction on $k = k_1 + \ldots + k_n$. If $k = 1$, then $\gamma = \gamma_1$ and $0 = \partial\gamma_1 = \gamma_1(b) - \gamma_1(a)$ by definition, hence $\gamma_1(b) = \gamma_1(a)$. In other words, γ_1 is a loop.

Assume the lemma for all $k \leq K - 1$, and suppose that $k_1 + \ldots + k_n = K$. If $\partial\gamma_1 = 0$, then

$$\gamma - k_1\gamma_1 = k_2\gamma_2 + \ldots + k_n\gamma_n,$$

is a cycle with $k_2 + \ldots + k_n = K - k_1 < K$. If $\partial\gamma_1 \neq 0$, then in order for $\partial\gamma = 0$ to be true, the terminal point of γ_1 must coincide with the initial point of at least one of $\gamma_2, \ldots, \gamma_n$. Without loss of generality, let γ_2 be such a curve, and define $\gamma_0 = \gamma_1 + \gamma_2$ which forms a genuine oriented curve. With this new curve γ can be written as

$$\gamma = \gamma_0 + (k_1 - 1)\gamma_1 + (k_2 - 1)\gamma_2 + \ldots + k_n\gamma_n,$$

and the sum of the coefficients is $1 + (k_1 - 1) + (k_2 - 1) + k_3 + \ldots + k_n = K - 1$. □

Corollary 24. *A connected open set $\Omega \subseteq \mathbb{C}$ is simply connected if and only if every cycle in Ω is null-homologous.*

[2]This claim is not used in these notes, except in §7 to motivate a couple of definitions. A related assumption will be used in Theorems 27 and 28, but it will be explicitly mentioned whenever there is an assumption involved. To prove the claim, one would have to show that the group $H_1(\Omega)$ has no torsion, i.e., that there is no element $[\gamma] \in H_1(\Omega)$ such that $n[\gamma] = 0$ for some $n \in \mathbb{N}$. For bounded open subsets of S^2 with finitely many holes, the homology group is torsion-free. Examples of compact manifolds with nonzero torsion in their homology group include nonorientable surfaces such as the Klein bottle and the real projective plane.

Proof. Let Ω be simply connected, and let γ be a cycle in Ω. The preceding lemma gives

$$\gamma = k_1\gamma_1 + \ldots + k_n\gamma_n,$$

where all γ_j are closed oriented curves. But $\gamma_j \approx 0$ by simple connectedness, meaning that γ is null-homologous. Since every loop is also a cycle, the other direction is trivial. $\square$

Corollary 25. *If Ω is simply connected then $\langle f, \gamma\rangle = 0$ for $f \in \mathcal{O}(\Omega)$ and for piecewise differentiable $\gamma \in Z_1(\Omega)$.*

7. The residue theorem

Let us assume that the homology group $H_1(\Omega)$ has a finite basis $[\gamma_1], \ldots, [\gamma_n]$ of n elements, i.e., that any homology class $[\gamma] \in H_1(\Omega)$ has a unique expansion

$$[\gamma] = k_1[\gamma_1] + \ldots + k_n[\gamma_n].$$

An equivalent way of saying this is that any cycle $\gamma \in Z_1(\Omega)$ can be written uniquely as

$$\gamma \sim k_1\gamma_1 + \ldots + k_n\gamma_n.$$

Then the integral of $f \in \mathcal{O}(\Omega)$ over γ can be computed as

$$\langle f, \gamma\rangle = k_1\langle f, \gamma_1\rangle + \ldots + k_n\langle f, \gamma_n\rangle,$$

meaning that once the integrals $\langle f, \gamma_j\rangle$ are computed, integration over general cycles reduces to knowing the coefficients k_j. For open sets $\Omega \subset \mathbb{C}$ with finitely many holes, it is known that the basis elements $[\gamma_j]$ can be chosen to correspond to loops around the holes in Ω. In particular the rank n of $H_1(\Omega)$, also called the first Betti number, is equal to the number of holes in Ω. So given a bounded domain with finite number of holes, and given a holomorphic function, in certain sense the only nontrivial integrals over cycles are the integrals over the contours around each hole.

We consider here the special situation where the relevant holes are isolated points. Let $\Omega \subset \mathbb{C}$ be a bounded open set, and let $z_1, \ldots, z_n \in \Omega$. Suppose that $\gamma \in B_1(\Omega)$ is a null-homologous cycle (in Ω) that does not intersect any of the points $z_1, \ldots, z_n$. Then in light of the above discussion, let us assume that as an element of $Z_1(\Omega \setminus \{z_1, \ldots, z_n\})$, the cycle γ can be written uniquely as[3]

$$\gamma \sim k_1\gamma_1 + \ldots + k_n\gamma_n, \qquad \text{with} \qquad \gamma_j = \partial D_\varepsilon(z_j), \quad j = 1, \ldots, n, \tag{2}$$

where $\varepsilon > 0$ is chosen so small that the circles $\partial D_\varepsilon(z_j)$ are pairwise disjoint and that each of those circles is disjoint from $\partial\Omega$. An approach to finding the coefficients k_j would be to construct functions g_ℓ, $(\ell = 1, \ldots, n)$, such that $\langle g_\ell, \gamma_j\rangle = \delta_{\ell j}$. Then it is easy to see that $k_j = \langle g_j, \gamma\rangle$. In trying to find such functions, let us note that for $a, b \in \mathbb{C}$ with $|a - b| > \varepsilon$

$$\int_{\partial D_\varepsilon(a)} \frac{\mathrm{d}z}{z-a} = 2\pi i, \qquad \text{and} \qquad \int_{\partial D_\varepsilon(a)} \frac{\mathrm{d}z}{z-b} = 0.$$

So if we define the *Cauchy kernel* K_a by $K_a(z) = \frac{1}{z-a}$, then we have $\langle K_{z_j}, \partial D_\varepsilon(z_\ell)\rangle = 2\pi i\delta_{j\ell}$, giving the formula

$$\langle f, \gamma\rangle = \frac{1}{2\pi i}\langle K_{z_1}, \gamma\rangle\langle f, \partial D_\varepsilon(z_1)\rangle + \ldots + \frac{1}{2\pi i}\langle K_{z_n}, \gamma\rangle\langle f, \partial D_\varepsilon(z_n)\rangle.$$

for arbitrary $f \in \mathcal{O}(\Omega \setminus \{z_1, \ldots, z_n\})$. This motivates the following definitions.

[3]This can be proven either by a purely topological argument, or by a careful study of isolated singularities as we will do in the next set of notes (in particular the reasoning will not be circular). Nevertheless, we will make it clear in the following if any particular assertion depends on this assumption.

Definition 26. For a cycle $\gamma \in Z_1(\mathbb{C})$ that does not pass through $a \in \mathbb{C}$, the *index* (or the *winding number*) of γ with respect to a is defined to be

$$\operatorname{Ind}(\gamma, a) = \frac{1}{2\pi i}\langle K_a, \gamma\rangle \equiv \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-a}.$$

If f is holomorphic on the punctured disk $D_r(a) \setminus \{a\}$ with $r > 0$, then with $\varepsilon \in (0, r)$

$$\operatorname{Res}(f, a) = \frac{1}{2\pi i}\langle f, \partial D_\varepsilon(a)\rangle \equiv \frac{1}{2\pi i}\int_{\partial D_\varepsilon(a)} f(z)\mathrm{d}z,$$

is called the *residue* of f at a. Note that the residue does not depend on the value of $\varepsilon \in (0, r)$, so in particular one could take the limit $\varepsilon \to 0$.

Modulo the assumption in (2), we have proved the following theorem.

Theorem 27. *Let $\Omega \subset \mathbb{C}$ be a bounded open set, and let $z_1, \ldots, z_n \in \Omega$. Suppose that $\gamma \in B_1(\Omega)$ is a null-homologous cycle that does not pass through any of the points $z_1, \ldots, z_n$. Then we have*

$$\langle f, \gamma\rangle = 2\pi i \sum_{j=1}^n \operatorname{Ind}(\gamma, z_j) \operatorname{Res}(f, z_j), \qquad \textit{for} \qquad f \in \mathcal{O}(\Omega \setminus \{z_1, \ldots, z_n\}).$$

8. Cauchy integral formulæ

If f is holomorphic at a, then $\operatorname{Res}(f, a) = 0$, meaning that in order to have a nonzero residue, f has to have a singularity at a. Assuming that f is holomorphic in a neighbourhood of a, let us compute the residue of $g = K_a f$ at a. For $z \in \partial D_\varepsilon(a)$ with small $\varepsilon > 0$, we have

$$g(z) = \frac{f(z)}{z-a} = \frac{f(a)}{z-a} + f'(a) + o(1),$$

inferring that

$$\langle g, \partial D_\varepsilon(a)\rangle = f(a)\langle K_a, \partial D_\varepsilon(a)\rangle + f'(a)\langle 1, \partial D_\varepsilon(a)\rangle + o(\varepsilon) = 2\pi i f(a),$$

where we have taken the limit $\varepsilon \to 0$ to conclude that the $o(\varepsilon)$ term is indeed zero. Hence

$$\operatorname{Res}(K_a f, a) = f(a), \tag{3}$$

and by applying Theorem 27 to $K_a f$, we immediately get the following result.

Theorem 28. *Let $\Omega \subset \mathbb{C}$ be a bounded open set, and let $a \in \Omega$. Suppose that $\gamma \in B_1(\Omega)$ is a null-homologous cycle that does not pass through a. Then for $f \in \mathcal{O}(\Omega)$ we have*

$$\operatorname{Ind}(\gamma, a) f(a) = \frac{1}{2\pi i}\langle K_a f, \gamma\rangle \equiv \frac{1}{2\pi i}\int_\gamma \frac{f(z)\mathrm{d}z}{z-a},$$

The main message of this result is that the values of f at points a with $\operatorname{Ind}(\gamma, a) \neq 0$ are determined by the values of f at the curve γ. The condition $\operatorname{Ind}(\gamma, a) \neq 0$ means that γ wraps around a, or that a is "inside" γ. Note that since we have used Theorem 27, this result depends on the assumption in (2). We will revisit Theorem 27 and give it a self-contained proof in the next set of notes, so for the sake of clarity in what follows we shall not use Theorems 27 and 28 anymore. To develop the theory further, we will need the following special case of Theorem 28, which we prove without the assumption in (2).

Theorem 29. *Let $f \in \mathcal{O}(\Omega)$, and let $\overline{D}_\varepsilon(a) \subset \Omega$ with $\varepsilon > 0$. Suppose that $\gamma \in Z_1(\Omega \setminus \{a\})$ is a cycle that is homologous to $\partial D_\varepsilon(a)$ in $\Omega \setminus \{a\}$. Then one has*

$$f(a) = \frac{1}{2\pi i}\langle K_a f, \gamma\rangle \equiv \frac{1}{2\pi i}\int_\gamma \frac{f(z)\mathrm{d}z}{z-a}.$$

Proof. Since $K_a f \in \mathcal{O}(\Omega \setminus \{a\})$, by Cauchy's theorem for homologous cycles we have

$$\langle K_a f, \gamma\rangle = \langle K_a f, \partial D_\varepsilon(a)\rangle = 2\pi i \mathrm{Res}(K_a f, a),$$

the latter equality by definition, and taking into account (3) the proof is established. □

9. The Cauchy-Taylor theorem

In this section we will complete the proof of Theorem 1 on page 1, by proving the implications (d) ⇒ (e) and (c) ⇒ (a).

Theorem 30. *Let $f \in C(\Omega)$, and assume that for all closed disks $\overline{D}_r(c) \subset \Omega$ one has*

$$f(\zeta) = \frac{1}{2\pi i}\langle K_\zeta f, \partial D_r(c)\rangle, \qquad \text{for} \qquad \zeta \in D_r(c).$$

Let $\overline{D}_r(c) \subset \Omega$ be any of those disks. Then the power series $f(\zeta) = \sum a_n(\zeta - c)^n$ with

$$a_n = \frac{1}{2\pi i}\langle K_c^{n+1} f, \partial D_\varepsilon(c)\rangle \equiv \frac{1}{2\pi i}\int_{\partial D_\varepsilon(c)} \frac{f(z)\mathrm{d}z}{(z-c)^{n+1}}, \qquad 0 < \varepsilon < r,$$

converges on $D_r(c)$. In particular, $\mathcal{O}(\Omega) \subseteq C^\omega(\Omega)$.

Proof. Without loss of generality, let us assume $c = 0$, and start with the integral formula

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{f(z)\mathrm{d}z}{z-\zeta}, \qquad \text{for} \quad \zeta \in D_\varepsilon,$$

with $\varepsilon \in (0, r)$. This can be rewritten as

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \left(\sum_{n=0}^{\infty} \frac{f(z)\zeta^n}{z^{n+1}}\right) \mathrm{d}z, \tag{4}$$

where we have used

$$\frac{1}{z-\zeta} = \frac{1}{z}\cdot\frac{1}{1-\zeta/z} = \frac{1}{z}\left(1 + \frac{\zeta}{z} + \dots\right) = \sum_{n=0}^{\infty}\frac{\zeta^n}{z^{n+1}}.$$

Each term in the series under integral in (4) can be estimated as

$$\left|\frac{f(z)\zeta^n}{z^{n+1}}\right| \le \frac{\|f\|_{D_\varepsilon}}{\varepsilon}\cdot\left(\frac{|\zeta|}{\varepsilon}\right)^n,$$

so as a function of z, the series converges uniformly on ∂D_ε. Therefore we can interchange the integral with the sum, resulting in

$$f(\zeta) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\zeta^n \int_{\partial D_\varepsilon}\frac{f(z)\mathrm{d}z}{z^{n+1}}.$$

Now the individual term of the series satisfies

$$\left|\zeta^n \int_{\partial D_\varepsilon}\frac{f(z)\mathrm{d}z}{z^{n+1}}\right| \le 2\pi\|f\|_{D_\varepsilon}\left(\frac{|\zeta|}{\varepsilon}\right)^n,$$

implying that the series converges locally normally in D_ε.

Finally, we note that since this proof implies that $f \in C^\omega(\Omega)$ so in particular f is holomorphic, the coefficients a_n do not depend on the choice of ε, and in particular $a_n = \mathrm{Res}(K_c^{n+1} f, c)$, since $K_c^{n+1} f$ is holomorphic on the punctured disk $D_r(c) \setminus \{c\}$. □

The following result gives a useful criterion to recognize holomorphy.

Theorem 31 (Morera's theorem)**.** *A function $f \in C(\Omega)$ is holomorphic if either*

(a) f is locally integrable, or

(b) $\langle f, \partial\tau\rangle = 0$ *for any closed triangle* $\tau \subset \Omega$*.*

Proof. Part (b) follows from (a) by Theorem 8 on page 4. For (a), by definition each point in Ω has a neighbourhood U and $F \in \mathcal{O}(U)$ such that $F' = f$ on U. The Cauchy-Taylor theorem guarantees that $F \in C^\omega(U)$, and by termwise differentiating we infer $f \in C^\omega(U)$. This means that $f \in C^\omega(\Omega)$, or in other words $f \in \mathcal{O}(\Omega)$. $\square$

As an application, one can prove that the locally uniform limit of holomorphic functions is holomorphic. This is to be contrasted with the situation in the real differentiable case where the uniform limit of smooth functions is not smooth in general.

Theorem 32 (Weierstrass convergence theorem)**.** *Let* $\{f_k\} \subset \mathcal{O}(\Omega)$ *be a sequence such that* $f_k \to f$ *locally uniformly on* Ω *for some* $f : \Omega \to \mathbb{C}$*. Then* $f \in \mathcal{O}(\Omega)$ *and* $f_k^{(n)} \to f^{(n)}$ *locally uniformly on* Ω*, for each* $n \in \mathbb{N}$*.*

Proof. First of all we have $f \in C(\Omega)$. Let $\tau \subset \Omega$ be a closed triangle. Then since $\partial\tau$ is compact, f_n converges uniformly to f on $\partial\tau$, and so we have

$$\langle f, \partial\tau\rangle = \lim_{k\to\infty} \langle f_k, \partial\tau\rangle = 0,$$

implying that $f \in \mathcal{O}(\Omega)$ by Morera's theorem.

For the second part of the claim we employ the Cauchy estimates. Let $D_{2\delta}(a) \subset \Omega$ for some $\delta > 0$. Then since $f_k - f \in \mathcal{O}(\Omega)$, for $k \in \mathbb{N}$ the Cauchy estimate gives

$$\|f_k^{(n)} - f^{(n)}\|_{D_\delta(a)} \leq \frac{C_n}{\delta^n}\|f_k - f\|_{D_{2\delta}(a)},$$

where $C_n > 0$ are constants. This completes the proof. $\square$

10. Simple connectivity

We list here several important consequences of simple connectivity. They will turn into characterizations of simple connectivity upon proving the Riemann mapping theorem, which states that for every connected open proper subset Ω of $\mathbb{C}$ satisfying the condition (g) of the following theorem, there is a bijection $\psi : \Omega \to \mathbb{D}$ such that $\psi \in \mathcal{O}(\Omega)$ and $\psi^{-1} \in \mathcal{O}(\mathbb{D})$. Here we denote by $\mathbb{D}$ the unit open disk in $\mathbb{C}$.

Theorem 33. *Let* $\Omega \subseteq \mathbb{C}$ *be an open set. Then each of the following statements (except (f)) implies the statement following it.*

(a) $\Omega \subseteq \mathbb{C}$ *is homeomorphic to* $\mathbb{D}$*.*
(b) Ω *is simply connected.*
(c) For any $f \in \mathcal{O}(\Omega)$*, and for any* $\gamma \in C^1_{\mathrm{pw}}(S^1, \Omega)$*, it holds that* $\langle f, \gamma\rangle = 0$*.*
(d) For any $f \in \mathcal{O}(\Omega)$*, there exists* $F \in \mathcal{O}(\Omega)$ *such that* $F' = f$*.*
(e) For any $f \in \mathcal{O}(\Omega)$ *with* $1/f \in \mathcal{O}(\Omega)$*, there is* $g \in \mathcal{O}(\Omega)$ *such that* $\exp \circ g = f$ *in* Ω*.*
(f) For any $f \in \mathcal{O}(\Omega)$ *with* $1/f \in \mathcal{O}(\Omega)$*, and for any integer* $n \geq 1$*, there is* $g \in \mathcal{O}(\Omega)$ *such that* $g(z)^n = f(z)$ *for all* $z \in \Omega$*.*

Proof. The implication (b) $\Rightarrow$ (c) is Corollary 20 on page 7, and (c) $\Rightarrow$ (d) is Theorem 10 on page 5. Moreover, (d) $\Rightarrow$ (e) can be proven by considering a primitive of f'/f on Ω, and then (e) $\Rightarrow$ (f) is immediate.

So we need only to prove (a) $\Rightarrow$ (b). Suppose that $\psi : \Omega \to \mathbb{D}$ is a homeomorphism. Obviously Ω is path-connected. Now if $\gamma : S^1 \to \Omega$ is a loop in Ω, then $\Gamma(t, s) = \psi^{-1}(s\psi(\gamma(t)))$ with $(t, s) \in S^1 \times [0, 1]$ is a homotopy of loops between the point $\psi^{-1}(0) \in \Omega$ (considered as a constant loop) and γ, hence Ω is simply connected. $\square$