# MATH 566 LECTURE NOTES 2: POWER SERIES, ANALYTICITY AND ELEMENTARY FUNCTIONS

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## 1. Uniform and normal convergences

For a set $\Omega \subseteq \mathbb{C}$ and a function $f : \Omega \to \mathbb{C}$, we define the *uniform norm*

$$\|f\|_\Omega = \sup_\Omega |f|.$$

We say that a sequence $\{f_k\}$ of functions $f_k : \Omega \to \mathbb{C}$ *converges uniformly* in $\Omega$ to $f$, if $\|f_k - f\|_\Omega \to 0$ as $k \to \infty$. Recall that this is equivalent to the sequence $\{f_k\}$ being a Cauchy sequence, i.e., $\|f_k - f_\ell\|_\Omega \to 0$ as $k, \ell \to \infty$.

**Theorem 1** (Weierstrass M-test)**.** *If $g_n : \Omega \to \mathbb{C}$ and $\sum_n \|g_n\|_\Omega < \infty$, then the series $\sum_n g_n$ converges uniformly in $\Omega$.*

*Proof.* With $f_k = \sum_{n\leq k} g_n$, we have for $\ell < k$

$$\|f_k - f_\ell\|_\Omega = \|\sum_{\ell<n\leq k} g_n\|_\Omega \leq \sum_{\ell<n\leq k} \|g_n\|_\Omega \leq \sum_{n>\ell} \|g_n\|_\Omega,$$

which tends to 0 when $\ell \to \infty$. So $\{f_k\}$ is a Cauchy sequence, hence converges. □

*Example* 2. The geometric series $\sum_n z^n$ converges uniformly in $D_r$ (and also in $\bar{D}_r$) as long as $r < 1$. To be pedantic, with $r < 1$ and with functions $g_n : D_r \to \mathbb{C}$ given by $g_n(z) = z^n$, the series $\sum_n g_n$ converges uniformly in $D_r$. However, the convergence is not uniform in the open unit disk $D_1$, and $\sum_n z^n$ does not converge if $|z| \geq 1$.

Convergence behaviour of frequently occurring sequences in complex analysis can be captured conveniently by the notion of locally uniform convergence. Recall that $U \subseteq \Omega$ is a *neighbourhood* of $z \in \Omega$ *in* $\Omega$ if there is an open set $V \subseteq \mathbb{C}$ such that $z \in V \cap \Omega \subseteq U$.

**Definition 3.** A sequence $\{f_k\}$ of functions $f_k : \Omega \to \mathbb{C}$ is called *locally uniformly convergent* in $\Omega$ if for each $z \in \Omega$ there is a neighbourhood $U \ni z$ in $\Omega$ such that $\{f_k\}$ converges uniformly in $U$.

*Example* 4. The series $\sum_n z^n$ converges locally uniformly in $D_1$.

Continuity is preserved under locally uniform convergence. In this regard uniform continuity is a very strong type of convergence; it naturally preserves uniform continuity.

**Theorem 5.** *If $\{f_k\} \subset C(\Omega)$ converges to $f : \Omega \to \mathbb{C}$ locally uniformly, then $f \in C(\Omega)$.*

*Proof.* Let $z \in \Omega$, and let $U \ni z$ be an open neighbourhood such that $\|f - f_k\|_U \to 0$. Let $\varepsilon > 0$. Then choose $\varrho > 0$ such that $B_\varrho(z) \subset U$, and $k$ such that $\|f - f_k\|_{B_\varrho(z)} \leq \varepsilon$. Moreover, let $\delta > 0$ be such that $|f_k(z) - f_k(w)| \leq \varepsilon$ whenever $w \in B_\delta(z)$. Then it is clear that $|f(z) - f(w)| \leq 3\varepsilon$ whenever $w \in B_\delta(z) \cap B_\varrho(z)$. □

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One drawback of (locally) uniform convergence is that a rearrangement of a uniformly convergent series can be divergent or converge to a limit different than the original series. So uniform convergence can be compared to conditional convergence of series of numbers. An improvement of uniform convergence that can be compared to absolute convergence of series of numbers is the notion of normal convergence.

**Definition 6.** A series $\sum_n g_n$ of functions $g_n : \Omega \to \mathbb{C}$ is called *normally convergent* in $\Omega$ if $\sum_n \|g_n\|_\Omega < \infty$. Moreover, we say $\sum_n g_n$ converges *locally normally* in $\Omega$ if for each $z \in \Omega$ there is an open neighbourhood $U \ni z$ such that $\sum_n \|g_n\|_U < \infty$.

The Weierstrass M-test immediately implies that every (locally) normally convergent series is (locally) uniformly convergent. It is also obvious from the definition that any subseries of a (locally) normally convergent series is (locally) normally convergent.

**Theorem 7.** *Let $f_{k,\ell} : \Omega \to \mathbb{C}$ for $k, l \in \mathbb{N}$, and let $\sigma : \mathbb{N}^2 \to \mathbb{N}$ be a bijection. Define the sequence $\{g_n\}$ by $g_{\sigma(k,\ell)} = f_{k,\ell}$. Then the followings are equivalent:*

*(a) The series $\sum_n g_n$ is locally normally convergent.*
*(b) For each $z \in \Omega$ there is an open neighbourhood $U \ni z$ such that the series $\sum_k(\sum_\ell \|f_{k,\ell}\|_U)$ converges. In particular, for all $k \in \mathbb{N}$, the series $\sum_\ell f_{k,\ell}$ is locally normally convergent.*
*(c) For each $z \in \Omega$ there is an open neighbourhood $U \ni z$ such that the series $\sum_\ell(\sum_k \|f_{k,\ell}\|_U)$ converges. In particular, for all $\ell \in \mathbb{N}$, the series $\sum_k f_{k,\ell}$ is locally normally convergent.*

*If any (so all) of the above conditions is satisfied, then there holds that*

$$\sum_\ell (\sum_k f_{k,\ell}) = \sum_k (\sum_\ell f_{k,\ell}) = \sum_n g_n.$$

*Proof.* First we prove the implication (a) $\Rightarrow$ (b). Let $U \subset \Omega$ such that $N = \sum_n \|g_n\|_U < \infty$. This obviously implies that for all $k \in \mathbb{N}$, $M_k = \sum_\ell \|f_{k,\ell}\|_U < \infty$. Let $\varepsilon > 0$ and let $m_k$ be such that $\sum_{\ell > m_k} \|f_{k,\ell}\|_U \leq 2^{-k}\varepsilon$. So for any $m$ we have

$$\sum_{k \leq m} (\sum_\ell \|f_{k,\ell}\|_U) \leq \sum_{k\leq m} (\sum_{\ell \leq m_k} \|f_{k,\ell}\|_U) + 2\varepsilon \leq N + 2\varepsilon.$$

Now we shall prove that $g = \sum_n g_n$ is equal to $f = \sum_k(\sum_\ell f_{k,\ell})$. To this end, let $m$ be such that $\sum_{k>m}(\sum_\ell \|f_{k,\ell}\|_U) \leq \varepsilon$, and let $\tilde{f}_\varepsilon = \sum_{k\leq m}\sum_{\ell \leq m_k} f_{k,\ell}$. Then we have

$$\|f - \tilde{f}_\varepsilon\|_U \leq \sum_{k>m}(\sum_\ell \|f_{k,\ell}\|_U) + \sum_{k\leq m}(\sum_{\ell > m_k} \|f_{k,\ell}\|_U) \leq 3\varepsilon.$$

Similarly, for sufficiently large $p$, the partial sum $\tilde{g}_p = \sum_{n \leq p} g_n$ satisfies

$$\|\tilde{g}_p - \tilde{f}_\varepsilon\|_U \leq \sum_{k>m}(\sum_\ell \|f_{k,\ell}\|_U) + \sum_{k\leq m}(\sum_{\ell > m_k} \|f_{k,\ell}\|_U) \leq 3\varepsilon,$$

and so we have

$$\|f - g\|_U \leq \|g - \tilde{g}_p\|_U + 6\varepsilon.$$

Since $\tilde{g}_p \to g$ and $\varepsilon$ is arbitrary, we conclude $f = g$.

For the other direction (b) $\Rightarrow$ (a), we start with the definition of $M_k$ and the condition $M = \sum_k M_k < \infty$ for a suitable $U$. Then we have for any $p$

$$\sum_{n \leq p} \|g_n\|_U \leq \sum_{k \leq m} \sum_{\ell \leq m} \|f_{k,\ell}\|_U \leq M,$$

where $m$ is such that $\{\ell \leq m\}^2 \supseteq \sigma^{-1}(\{n \leq p\})$. The equivalence of (a) and (c) can be proven analogously. $\square$

**Corollary 8.** *Suppose that the series $\sum_n g_n$ converges locally normally in $\Omega$ to $g$. Let $\mathbb{N} = \cup_k M_k$ be a disjoint decomposition of $\mathbb{N}$, where $k$ is from a countable set and each $M_k = \{m_{k,\ell}\}$ is countable. Then for each $k$, the series $\tilde{g}_k = \sum_\ell g_{m_{k,\ell}}$ converges locally normally in $\Omega$, and moreover the series $\sum_k \tilde{g}_k$ converges locally normally in $\Omega$ to $g$.*

*Proof.* Let us say $k$ runs over $K \subseteq \mathbb{N}$, and for each $k \in K$, $\ell$ runs over $L_k \subseteq \mathbb{N}$. Define the set $M \subseteq \mathbb{N}^2$ by $M = \{(k,\ell) : k \in K \text{ and } \ell \in L_k\}$, and let $f_{k,\ell} = g_{m_{k,\ell}}$ if $(k,\ell) \in M$, and $f_{k,\ell} = 0$ otherwise. Then we get the proof by applying the above theorem with, e.g., $\sigma(k,\ell) = 2m_{k,\ell}$ for $(k,\ell) \in M$, and $\sigma(k,\ell) = 2\tau(k,\ell) - 1$ for $(k,\ell) \in \mathbb{N}^2 \setminus M$, where $\tau : \mathbb{N}^2 \setminus M \to \mathbb{N}$ is a bijection. □

The above corollary with each $M_k$ having a single element gives the following rearrangement result.

**Corollary 9.** *Suppose that the series $\sum_n g_n$ converges locally normally in $\Omega$ to $g$. Then given any bijection $\tau : \mathbb{N} \to \mathbb{N}$, the rearranged series $\sum_n g_{\tau(n)}$ also converges locally normally in $\Omega$ to $g$.*

**Corollary 10.** *If $f = \sum_m f_m$ and $g = \sum_n g_n$ are locally normally convergent series in $\Omega$, then for every bijection $\sigma : \mathbb{N}^2 \to \mathbb{N}$, the series $\sum_p h_p$ with elements $h_{\sigma(m,n)} = f_m g_n$ converges locally normally in $\Omega$ to $fg$.*

*Proof.* For any $z \in \Omega$ we can certainly find a neighbourhood $U \ni z$ such that

$$\sum_\ell \|f_k g_\ell\|_U \leq \|f_k\|_U \sum_\ell \|g_\ell\|_U < \infty,$$

and that

$$\sum_k (\sum_\ell \|f_k g_\ell\|_U) \leq \sum_k \|f_k\|_U (\sum_\ell \|g_\ell\|_U) = (\sum_k \|f_k\|_U)(\sum_\ell \|g_\ell\|_U) < \infty.$$

Note that these imply

$$\sum_\ell f_k g_\ell = f_k g, \qquad \text{and} \qquad \sum_k (\sum_\ell f_k g_\ell) = fg.$$

The proof is established upon employing Theorem 7 (b) with $f_{k,\ell} = f_k g_\ell$. □

*Remark* 11. Apart from "local" convergences, one can talk about "compact" convergences, which require a convergence in all compact subsets of $\Omega$. Examples include compactly uniform convergence (which is often simply called compact convergence) and compactly normal convergence. For subsets of the complex plane, it can be shown by covering arguments that these compact convergences are equivalent to their local counterparts (In fact the equivalence holds in any locally compact topological space). Another frequently occurring term is "absolute convergence", which means that the real valued function $|f| : \Omega \to \mathbb{R}$ enters the discussion. For example, "uniformly absolute convergence" designates the uniform convergence of the function series $\sum_n |f_n|$. Note that this is in contrast to the normal convergence where we consider the series $\sum_n \|f_n\|_\Omega$ of numbers. In particular, normal convergence implies uniformly absolute convergence, but the converse is not true as can be seen from a sequence of bumps of height $1/n$ going off to infinity. The local version of the above "uniformly absolute convergence" is "locally uniformly absolute convergence"; one now can guess what it should mean. It is instructive to try to prove Theorem 7 and its corollaries for locally uniformly absolutely converging series.

*Remark* 12. There seems to be no universally used "unambiguous language" in literature concerning the terminology on various modes of convergence. For instance, depending on which book you read, "normal convergence" may mean "locally uniform convergence" or "locally normal convergence", and "absolute convergence" may mean "uniform absolute convergence" or "pointwise absolute convergence".

## 2. Power series

We understand by a *power series* an expression of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n, \tag{1}$$

with the coefficients $a_n \in \mathbb{C}$, and the centre $c \in \mathbb{C}$. Assume that the above series converges. Then obviously $|a_n||z-c|^n \to 0$ as $n \to \infty$, so that $|a_n||z-c|^n \leq M$ for some constant $M < \infty$. In other words, we have

$$|z-c| \leq R := \sup\{r \geq 0 : \sup_n |a_n| r^n < \infty\}. \tag{2}$$

Put another way, the power series (1) diverges whenever $|z-c| > R$. The converse statement is almost true as seen from the theorem below, which justifies the fact that the above defined $R \in [0,\infty]$ is called the *convergence radius* of the power series (1).

**Theorem 13.** *Let $R \in [0,\infty]$ be defined by* (2)*. Then the power series* (1) *converges locally normally in the open disk $D_R(c)$, and diverges at every $z \in \mathbb{C} \setminus \bar{D}_R(c)$. Moreover, $R$ can be determined by the* Cauchy-Hadamard formula

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}, \tag{3}$$

*with the conventions $1/\infty = 0$ and $1/0 = \infty$, and furthermore, provided that $a_n = 0$ for only finitely many $n$, one can estimate $R$ by the* ratio test

$$\liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \leq R \leq \limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}. \tag{4}$$

*Proof.* Without loss of generality, we may take $c = 0$. Divergence at every $z \in \mathbb{C} \setminus \bar{D}_R(c)$ is demonstrated above. For convergence, let $z \in D_r$ with $r < R$. Then for any $\rho \in (r,R)$ we have $|a_n z^n| \leq |a_n|\rho^n \frac{r^n}{\rho^n} \leq M\frac{r^n}{\rho^n}$ for some constant $M < \infty$. Since $\frac{r}{\rho} < 1$, $\sum a_n z^n$ converges normally in $D_r$. Since any $z \in D_R$ is in some such $D_r$ with $r < R$, the series converges locally normally in $D_R$.

To prove (3), let $\varrho$ be defined by $1/\varrho = \limsup_{n\to\infty} |a_n|^{1/n}$ with the intention of showing that $\varrho = R$. By definition, for any $\varepsilon \in (0,1)$, we have $|a_n|\varrho^n \geq (1-\varepsilon)^n$ for infinitely many $n$, and there is $n_\varepsilon$ such that $|a_n|\varrho^n \leq (1+\varepsilon)^n$ for all $n > n_\varepsilon$. Thus if $|z| > \varrho$ then $|a_n z^n| > 1$ for infinitely many $n$, and the series $\sum a_n z^n$ diverges. This implies that $\varrho \geq R$. On the other hand, if $|z| < \varrho$, then for any $\varepsilon > 0$ we have $|a_n z^n| \leq |a_n|\varrho^n \frac{|z|^n}{\varrho^n} \leq (1+\varepsilon)^n \frac{|z|^n}{\varrho^n} =: k^n$ for all $n > n_\varepsilon$. By choosing $\varepsilon > 0$ small enough, one can ensure that $k \in [0,1)$, and so $\sum a_n z^n$ converges. This implies that $\varrho \leq R$.

Now we shall prove the ratio test. Let $\alpha$ be the limit infimum in (4) and suppose that $|z| < \alpha$. By definition, for any $\varepsilon > 0$ we have $|a_n| \geq (\alpha-\varepsilon)|a_{n+1}|$ for all sufficiently large $n$. This gives $|a_n z^n| \leq C(\frac{|z|}{\alpha-\varepsilon})^n$ for all sufficiently large $n$, with some constant $C > 0$. By choosing $\varepsilon$ small enough we show the convergence of $\sum a_n z^n$, which implies that $\alpha \leq R$.

For the upper bound on $R$, let $\beta$ be the limit supremum in (4), and suppose that $|z| > \beta$ and $\varepsilon = |z| - \beta > 0$. Then by definition, we have $|a_n| \leq (\beta+\varepsilon)|a_{n+1}|$ for all sufficiently large $n$. So $|a_n z^n| \geq \frac{C|z|^n}{(\beta+\varepsilon)^n} \geq C$ for some constant $C > 0$, and the series diverges. This implies that $R \leq \beta$. $\square$

As an application, let us derive the product and quotient rules for power series.

**Corollary 14.** *Let $R > 0$ and $S > 0$ be the convergence radii of the power series $f(z) = \sum a_n(z-c)^n$ and $g(z) = \sum b_n(z-c)^n$, respectively. Then we have*

$$f(z)g(z) = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} a_j b_k \right) (z-c)^n,$$

*where the convergence radius of the power series is at least* $\min\{R, S\}$.

*Furthermore, if $b_0 \neq 0$, then*

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} e_n (z-c)^n, \qquad \text{with} \qquad e_n = \frac{1}{b_0} \left( a_n - \sum_{k=0}^{n-1} b_{n-k} e_k \right),$$

*where the power series converges in a neighbourhood of $c$, and the empty sum in the definition of $e_n$ when $n = 0$ is understood to be $0$.*

*Proof.* The product rule is from Corollary 10 and the above characterization of convergence radii. The recursive formula for $e_n$ can be formally derived from the product rule, namely from the formula

$$a_n = \sum_{k=0}^{n} b_{n-k} e_k = b_0 e_n + \sum_{k=0}^{n-1} b_{n-k} e_k.$$

For convergence of the quotient series, let $M > 0$ and $r > 0$ be constants such that $|a_n|$ and $|b_n|$ are both bounded by $Mr^{-n}|b_0|$. Then the definition of $e_n$ gives

$$|e_n| \leq Mr^{-n} + M \sum_{k=0}^{n-1} r^{k-n} |e_k|,$$

which, with the shorthand notation $K_n = |e_n| r^n$, maybe manipulated as

$$K_n \leq M + M \sum_{k=0}^{n-1} K_k \leq M(1+M)^n,$$

where the latter inequality can be proven by induction. This means that

$$|e_n| \leq M(1+M)^n r^{-n},$$

implying that the power series converges in the disk of radius $r/(1+M)$. □

Note that the convergence radius of the product series can actually be larger than $\min\{R, S\}$, because of possible cancellations in the sum $\sum a_j b_k$. Similarly, the following corollary gives only the worst case estimate on the convergence radius of the rearranged series, where the centre $c$ is moved to another point $d$ in the convergence disk.

**Corollary 15.** *Let $R > 0$ be the convergence radius of the power series $f(z) = \sum a_n(z-c)^n$, and let $d \in D_R(c)$. Then we have*

$$f(z) = \sum_{j=0}^{\infty} \left( \sum_{n=j}^{\infty} \binom{n}{j} a_n (d-c)^{n-j} \right) (z-d)^j,$$

*where the convergence radius of the power series is at least $R - |d-c|$. In particular, the convergence radius of a rearranged power series depends continuously on its centre.*

*Proof.* We have

$$(z-c)^n = (z-d+d-c)^n = \sum_{j=0}^{n} \binom{n}{j} (z-d)^j (d-c)^{n-j},$$

so that the proof is established upon justifying

$$\sum_{n=0}^{\infty} a_n \sum_{j=0}^{n} \binom{n}{j} (z-d)^j (d-c)^{n-j} = \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \binom{n}{j} a_n (z-d)^j (d-c)^{n-j},$$

for $z \in D_r(d)$ with $r = R - |d-c|$. This can be done by applying Corollary 8 if the left hand side is locally normally convergent in $D_r(d)$. To this end, let $|z-d| \leq \rho - |d-c|$ with $\rho < R$. Then we have

$$\sum_{j=0}^{n} \binom{n}{j} |z-d|^j |d-c|^{n-j} = (|z-d| + |d-c|)^n \leq \rho^n,$$

and since $a_n \rho^n = a_n R^n (\rho/R)^n$ we obtain the desired local normal convergence.

The continuity of convergence radius can be shown as follows. Let $R'$ denote the convergence radius of the rearranged series centred at $d$. We have $R' \geq R - |d-c|$, or put differently, $R - R' \leq |d-c|$. So if $|d-c| < R/2$ it is obvious that $c \in D_{R'}(d)$, which means that the above reasoning can be applied with the roles of the two power series interchanged, giving $R' - R \leq |c-d|$. □

Finally, we turn to the question of termwise differentiating and integrating power series. One consequence of this is that any power series is holomoprhic in its disk of convergence.

**Theorem 16.** *Let $R$ be the convergence radius of the power series* (1)*. Then both*

$$g(z) = \sum_{n=0}^{\infty} n a_n (z-c)^{n-1}, \qquad \text{and} \qquad F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-c)^{n+1},$$

*have convergence radii equal to $R$, and there hold that*

$$f' = g \qquad \text{and} \qquad F' = f, \qquad \text{in } D_R(c).$$

*Proof.* It is obvious that the convergence radius $R'$ of the power series representing $g$ is at most $R$, that is, $R' \leq R$. To prove the other direction, let $r < R$. Then for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$n|a_n|r^n \leq C_\varepsilon (1+\varepsilon)^n |a_n| r^n \leq C_\varepsilon (1+\varepsilon)^n (r/R)^n |a_n| R^n,$$

and choosing $\varepsilon$ small enough we see that $r \leq R'$, and so $R \leq R'$.

Now we will show that $f' = g$ in $D_R(c)$, i.e., that for each $z \in D_R(c)$ one has

$$f(z+h) = f(z) + g(z)h + o(|h|).$$

To this end, we write

$$f(z+h) - f(z) = \sum_{n=0}^{\infty} a_n \left((z+h)^n - z^n\right) = h \sum_{n=0}^{\infty} a_n \sum_{j=0}^{n-1} (z+h)^j z^{n-1-j} =: h\lambda_z(h).$$

Let $r < R$ be such that $|z| < r$, and consider all $h$ satisfying $|z+h| \leq r$. Then

$$\sum_{n=0}^{\infty} |a_n| \sum_{j=0}^{n-1} |z+h|^j |z|^{n-1-j} \leq \sum_{n=0}^{\infty} |a_n| n r^{n-1} < \infty,$$

so the series for $\lambda_z$ converges locally uniformly in a neighbourhood of the origin. Hence $\lambda_z$ is continuous at 0, and moreover from $\lambda_z(0) = g(z)$, we infer

$$\lambda_z(h) = g(z) + o(1),$$

with $o(1) \to 0$ as $|h| \to 0$. The claim is proven since

$$f(z+h) - f(z) = h(g(z) + o(1)) = hg(z) + o(|h|).$$

The claims about $F$ follow from the above if we start with $F$ instead of $f$. $\square$

## 3. Analyticity

In what follows, $\Omega$ will denote an open subset of $\mathbb{C}$.

**Definition 17.** A complex-valued function $f : \Omega \to \mathbb{C}$ is called *(complex) analytic at* $z \in \Omega$ if it is developable into a power series around $z$, i.e, if there are coefficients $a_n \in \mathbb{C}$ and a radius $r > 0$ such that the following equality holds for all $h \in D_r$

$$f(z+h) = \sum_{n=0}^{\infty} a_n h^n.$$

Moreover, $f$ is said to be *(complex) analytic on* $\Omega$ if it is analytic at each $z \in \Omega$. The set of analytic functions on $\Omega$ is denoted by $C^\omega(\Omega)$.

We observe from Corollary 14 that the product of two analytic functions is analytic, and that their quotient is analytic wherever the denominator function is nonzero. Also from Corollary 15 it is immediate that any power series is analytic in its disk of convergence. Moreover, Theorem 16 implies that analytic functions are holomorphic, and they are infinitely often differentiable: $C^\omega(\Omega) \subseteq \mathcal{O}(\Omega)$ and $C^\omega(\Omega) \subseteq C^\infty(\Omega)$. Actually, the converse $\mathcal{O}(\Omega) \subseteq C^\omega(\Omega)$ is also true, which we will prove in the next set of notes. Returning to the current set of affairs, and by repeatedly applying Theorem 16 we see that the coefficients of the power series of $f$ about $c \in \Omega$ are given by $a_n = f^{(n)}(c)/n!$, or in other words, if $f \in C^\omega(\Omega)$ and $c \in \Omega$ then the following *Taylor series* converges in a neighbourhood of $c$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(z-c)^n. \tag{5}$$

Recall that an *accumulation point* of a set $D \subset \mathbb{C}$ is a point $z \in \mathbb{C}$ such that any neighbourhood of $z$ contains a point $w \neq z$ from $D$. We say that $z \in D$ is an *isolated point* if it is not an accumulation point of $D$. If all points of $D$ are isolated $D$ is called *discrete*.

**Theorem 18** (Identity theorem)**.** *Let $f \in C^\omega(\Omega)$ with $\Omega$ a connected open set, and let the zero set of $f$ has an accumulation point in $\Omega$. Then $f \equiv 0$ in $\Omega$.*

*Proof.* Each $\Sigma_n = \{z \in \Omega : f^{(n)}(z) = 0\}$ is relatively closed in $\Omega$, so the intersection $\Sigma = \bigcap_n \Sigma_n$ is also closed. But $\Sigma$ is also open, because $z \in \Sigma$ implies that $f \equiv 0$ in a small disk centred at $z$ by a Taylor series argument. We shall prove below that $\Sigma$ is nonempty, which would conclude that $\Sigma = \Omega$.

Let $c \in \Omega$ be an accumulation point of $\Sigma_0$. If $c \in \Sigma$, then $\Sigma = \Omega$. If $c \notin \Sigma$, then there is $n$ such that $f^{(n)}(c) \neq 0$. So we have $f(z) = (z-c)^n g(z)$ for some continuous function $g$ with $g(c) \neq 0$. This will imply the existence of a neighbourhood of $c$ where $f$ has at most one zero, contradicting that $c$ is an accumulation point of the zero set of $f$. $\square$

The following corollary records the fact that an analytic function is completely determined by its restriction to any non-discrete subset of its domain of definition. In other words, if it is at all possible to extend an analytic function (defined on a non-discrete set) to a bigger domain, then there is only one way to do the extension.

**Corollary 19** (Uniqueness of analytic continuation)**.** *Let $u, v \in C^\omega(\Omega)$ with $\Omega$ a connected open set, and let $u \equiv v$ in a non-discrete set $D \subset \Omega$. Then $u \equiv v$ in $\Omega$.*

A powerful tool to perform analytic continuation is power series. There is a theorem to the effect that if an analytic continuation is at all possible, then power series can do it. We need the following estimates on power series coefficients, which roughly says that the largest term in the series determines the maximum absolute value the power series can have in a given region.

**Lemma 20** (Cauchy estimates). *Let $f(z) = \sum a_n z^n$ be a uniformly convergent power series in the closed disk $\bar{D}_\rho$. Then it holds that*

$$|a_n|\rho^n \leq \max_{|z|=\rho} |f(z)|, \qquad \text{for all } n.$$

*Proof.* We prove the lemma for $\rho = 1$, the general case follows by scaling. First we derive the claimed bound on $a_0$ in terms of $M = \max\limits_{|z|=1} |f(z)|$. Defining $p_m(z) = \sum_{n=0}^m a_n z^n$, we obviously have $p_m \to f$ uniformly on $\partial D_1$ as $m \to \infty$, so in particular $|p_m| < M + \varepsilon_m$ on the unit circle, with $\varepsilon_m \to 0$ as $m \to \infty$. If $\xi = \cos(\pi t) + i \sin(\pi t)$ with some irrational number $t$, then $\xi^k \neq 1$ whenever $k \neq 0$. So for any integer $\ell$ we have

$$\sum_{k=0}^{\ell-1} p_m(\xi^k) = \sum_{k=0}^{\ell-1}\sum_{n=0}^{m} a_n \xi^{kn} = \sum_{n=0}^{m} a_n \sum_{k=0}^{\ell-1} \xi^{kn} = \ell a_0 + \sum_{n=1}^{m} a_n \frac{\xi^{n\ell}-1}{\xi^n - 1}. \tag{6}$$

Let us estimate the rightmost term as follows

$$\left|\sum_{n=1}^{m} a_n \frac{\xi^{n\ell}-1}{\xi^n-1}\right| \leq \sum_{n=1}^{m} \frac{2|a_n|}{|\xi^n - 1|} =: \lambda_m,$$

where the real number $\lambda_m \geq 0$ is independent of $\ell$. As a result we get

$$|a_0| \leq \frac{1}{\ell}\sum_{k=0}^{\ell-1} |p_m(\xi^k)| + \frac{\lambda_m}{\ell} < M + \varepsilon_m + \frac{\lambda_m}{\ell},$$

and since $\varepsilon_m \to 0$ we conclude that $|a_0| \leq M$.

The bound on $a_n$ can be proven similarly, for instance, by summing over $-n \leq k \leq \ell - 1$ instead of $0 \leq k \leq \ell - 1$ in (6). □

An immediate consequence is the following quite remarkable theorem, which says that any Taylor series (with fixed centre) converges in the largest possible disk. Combined with Corollary 19, this reveals the possibility that power series can be used to extend analytic functions to larger domains than they are originally defined.

**Theorem 21.** *Let $f \in C^\omega(\Omega)$ and let $D_r(c) \subseteq \Omega$. Then the Taylor series of $f$ centred at $c$ converges in $D_r(c)$.*

*Proof.* We will prove that if the Taylor series of $f$ centred at 0 converges uniformly in the closed disk $\bar{D}_\rho$, and if $f$ is analytic in a neighbourhood of $\bar{D}_\rho$, then the convergence radius of the considered series is strictly larger than $\rho$. So termwise differentiating we have

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(0)}{(n-k)!} z^{n-k},$$

uniformly converging in $\bar{D}_\rho$. Then the previous lemma gives

$$\frac{|f^{(n)}(0)|}{(n-k)!}\rho^{n-k} \leq \max_{|w|=\rho} |f^{(k)}(w)|. \tag{7}$$

Since $f$ is analytic in a neighbourhood of $\bar{D}_\rho$, for every $w \in \partial D_\rho$ the Taylor series of $f$ centred at $w$ has convergence radius $R_w > 0$. Now by continuity of $w \mapsto R_w$ there is $\sigma > 0$ such that $\sigma < R_w$ for any $w \in \partial D_\rho$, meaning that the Taylor series centred at $w \in \partial D_\rho$ converges uniformly in $\bar{D}_\sigma(w)$ with $\sigma > 0$ independent of $w$. So another application of Lemma 20 gives, for any $w \in \partial D_\rho$

$$\frac{|f^{(k)}(w)|}{k!}\sigma^k \leq \max_{|z-w|=\sigma} |f(z)| \leq \max_{|z|\leq \rho+\sigma} |f(z)| =: M.$$

By combining this with (7) and summing over $k = 0, \ldots, n$, we infer

$$|f^{(n)}(0)|(\rho+\sigma)^n \leq M(n+1)!.$$

From this it is clear that the Taylor series

$$f(z) = \sum \frac{f^{(n)}(0)}{n!} z^n,$$

converges in $D_{\rho+\sigma}$. ☐

Liouville's theorem says that bounded entire analytic functions are constant. It is possible to devise a proof that uses Cauchy estimates (Lemma 20) directly, but the use of the above theorem gives a slight simplification.

**Corollary 22.** *If $f \in C^\omega(\mathbb{C})$, and if there exists a constant $M > 0$ such that $|f(z)| \leq M|z|^n$ for $z \in \mathbb{C} \setminus D_1$, then $f$ must be a polynomial of degree at most $n$.*

*Proof.* By the above theorem, the Taylor series of $f$ centred at the origin converges uniformly on any closed disk. Applying Lemma 20 to this series on $\bar{D}_\rho$ with $\rho > 1$, we get the following bound on the $k$-th coefficient

$$|a_k|\rho^k \leq \max_{|z|=\rho} |f(z)| \leq M\rho^n.$$

Since $\rho$ can be arbitrarily large, this estimate shows that $a_k = 0$ for all $k > n$. ☐

Liouville's theorem is the case $n = 0$ in the preceding corollary, and can be used to prove the fundamental theorem of algebra.

**Corollary 23.** *Any nonconstant polynomial has at least one root in $\mathbb{C}$.*

*Proof.* Suppose that a polynomial $p$ has no root. Then $f = \frac{1}{p} \in C^\omega(\mathbb{C})$. If $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ with $a_n \neq 0$, then $|p(z)| \sim |a_n||z|^n$ for large $z$, meaning that $f$ is bounded. By Liouville's theorem $f$ must be constant, contradicting the hypothesis. ☐

Lemma 20 can also be used to prove the open mapping theorem. We remark that the conditions of the following theorem can be slightly weakened so that $\Omega$ is not necessarily connected and $f$ is not constant on any of the connected components of $\Omega$.

**Theorem 24** (Open mapping theorem)**.** *Let $\Omega$ be a connected open set, and suppose that $f \in C^\omega(\Omega)$ is not a constant function. Then $f : \Omega \to \mathbb{C}$ is an open mapping, i.e., it sends open sets to open sets.*

*Proof.* Without loss of generality let us assume that $0 \in \Omega$ and that $f(0) = 0$. We will prove that a small disk centred at the origin will be mapped by $f$ to a neighbourhood of the origin. Let $D_r \subseteq \Omega$ with $r > 0$, and let $w \notin f(D_r)$. Then the function $\phi(z) = \frac{1}{f(z)-w}$ is analytic in $D_r$. Choose $0 < \rho < r$ so small that $f(z) = 0$ has no solution with $|z| = \rho$, so that $\delta = \inf_{|z|=\rho} |f(z)| > 0$. This is possible by the identity theorem since $f$ is not constant and $\Omega$ is connected. Since $\rho < r$, the Taylor series of $\phi$ about 0 converges uniformly in the closed disk $\bar{D}_\rho$. Now we apply Lemma 20 to $\phi$ and get

$$|\phi(0)| \leq \sup_{|z|=\rho} |\phi(z)| = \left(\inf_{|z|=\rho} |f(z)-w|\right)^{-1},$$

which, taking into account that $|\phi(0)| = |w|^{-1}$, is equivalent to

$$\inf_{|z|=\rho} |f(z) - w| \leq |w|.$$

We have $|f(z) - w| \geq |f(z)| - |w| \geq \delta - |w|$ for $|z| = \rho$, therefore the above estimate gives $|w| \geq \delta/2$. It follows that $D_{\delta/2} \subseteq f(D_r)$. □

Thus for example, one cannot get a (nonzero-length) curve as the image of a connected set under an analytic mapping. In particular, the only real-valued analytic functions defined on an open set in $\mathbb{C}$ are locally constant functions.

If $f$ is analytic, then the function $|f|^2$ is subharmonic, so the maximum principle for subharmonic functions apply to the modulus of an analytic function. Nevertheless, the open mapping theorem can be used to obtain a more insightful proof.

**Corollary 25** (Maximum principle). *Let $f \in C^\omega(\Omega)$ with $\Omega$ an open subset of $\mathbb{C}$.*

*(a) If $\Omega$ is connected and $|f(z)| = \sup_\Omega |f|$ at some $z \in \Omega$, then $f$ is constant.*

*(b) If $\Omega$ is bounded and $f \in C(\bar{\Omega})$, then we have $\sup_\Omega |f| \leq \max_{\partial\Omega} |f|$.*

*Proof.* The hypothesis in (a) says that $f(z)$ is a boundary point of the image $f(\Omega)$, since otherwise there would have to be a point in $f(\Omega)$ with absolute value strictly greater than $|f(z)|$. If $f$ is not a constant, by the open mapping theorem $f(\Omega)$ cannot include any of its boundary points, leading to a contradiction.

For part (b), there is $z \in \bar{\Omega}$ with $|f(z)| = \sup_{\bar{\Omega}} |f|$ since $\Omega$ is bounded and $f$ is continuous on $\bar{\Omega}$. If $z \in \partial\Omega$ then we are done; otherwise applying part (a) to the connected component of $\Omega$ that contains $z$ concludes the proof. □

We end this section with two simple corollaries of the open mapping theorem, the latter of which gives an alternative proof of the fundamental theorem of algebra.

**Corollary 26** (Preservation of domains). *If $\Omega \subseteq \mathbb{C}$ is connected open set and $f \in C^\omega(\Omega)$ nonconstant, then $f(\Omega)$ is also a connected open set.*

**Corollary 27.** *Let $f \in C^\omega(\mathbb{C})$, and suppose that $|f(z)| \to \infty$ as $|z| \to \infty$. Then $f(\mathbb{C}) = \mathbb{C}$.*

*Proof.* Obviously $f$ is not constant, and so the open mapping theorem implies that $f(\mathbb{C})$ is open. Let us show that $f(\mathbb{C})$ is also closed. Suppose that $f(z_k) \to w \in \mathbb{C}$ as $k \to \infty$. Then $\{z_k\}$ is bounded, so taking a subsequence if necessary, there is $z \in \mathbb{C}$ such that $z_k \to z$. By continuity $f(z_k) \to f(z)$, concluding that $w = f(z)$. □

## 4. Elementary functions

We look for the complex exponential as a solution of the following problem

$$f' = f, \qquad f(0) = 1,$$

and we look for it in the form of a power series. Formally termwise differentiating we find $a_n = a_{n-1}/n = \ldots = a_0/n!$, and the condition $f(0) = 1$ gives $a_0 = 1$. So the complex exponential is given by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{8}$$

which converges (e.g. by the ratio test) locally normally in $\mathbb{C}$, so in particular $\exp \in C^\omega(\mathbb{C})$.

For $a \in \mathbb{C}$, let $g(z) = \exp(z)\exp(a-z)$. Then $g'(z) = \exp(z)\exp(a-z) - \exp(z)\exp(a-z) = 0$ for all $z \in \mathbb{C}$, thus $g(z) \equiv g(0) = \exp(a)$. In other words, we have

$$\exp(z + w) = \exp(z)\exp(w), \qquad z, w \in \mathbb{C}. \tag{9}$$

Putting $w = -z$, we infer

$$\exp(-z)\exp(z) = 1 \qquad \text{and so} \qquad \exp(z) \neq 0 \qquad \forall z \in \mathbb{C}.$$

Similarly to the above, by considering $g(z) = f(z)\exp(-z)$, one can also show that the only function satisfying that $f' = f$ in $\mathbb{C}$ and $f(0) = 1$ is exp.

We can construct an analytic inverse of the exponential function in the open disk $D_1(1)$. Any such an inverse function is called a *logarithm function.*

**Lemma 28.** *Consider the power series*

$$\lambda(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n, \tag{10}$$

*whose convergence radius is* 1. *Then we have*

$$\exp\lambda(z) = z \qquad \text{and} \qquad \lambda'(z) = \frac{1}{z} \qquad \text{for} \quad z \in D_1(1).$$

*Proof.* Termwise differentiation gives

$$\lambda'(z) = \sum_{n=1}^{\infty}(-1)^{n-1}(z-1)^{n-1} = \sum_{n=1}^{\infty}(1-z)^{n-1} = \frac{1}{1-(1-z)} = \frac{1}{z},$$

provided that $|1-z| < 1$, that is, $z \in D_1(1)$.

Now let $g(z) = z\exp(-\lambda(z))$. Then for $z \in D_1(1)$ we have

$$g'(z) = \exp(-\lambda(z)) - z\exp(-\lambda(z))\lambda'(z) = 0,$$

meaning that $g(z) = g(1) = \exp(-\lambda(1)) = 1$ in $D_1(1)$. □

Let us turn to the question of identifying the range and the kernel of the group homomorphism $\exp : \mathbb{C} \to \mathbb{C}^\times$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is the multiplicative group of $\mathbb{C}$. Recall that the range is $\exp\mathbb{C} = \{\exp z : z \in \mathbb{C}\}$, the kernel is $\ker(\exp) = \{z \in \mathbb{C} : \exp z = 1\}$, and that they are subgroups of the additive group $\mathbb{C}$ and the multiplicative group $\mathbb{C}^\times$, respectively.

As a preliminary to the theorem that follows, let us investigate the solutions of the equation $|\exp z| = 1$. If $z = iy$ with $y \in \mathbb{R}$, then since $\overline{(iy)^n} = (-iy)^n$, formula (8) gives $\overline{\exp(iy)} = \exp(-iy)$. Hence $|\exp(iy)|^2 = \exp(iy)\overline{\exp(iy)} = 1$, meaning that $|\exp(z)| = 1$ whenever $\Re z = 0$. Now if $z = x+iy$ with $x, y \in \mathbb{R}$, then $|\exp z| = |\exp(x)\exp(iy)| = |\exp(x)||\exp(iy)| = \exp x$. To conclude,

$$|\exp z| = e^{\Re z}, \qquad \text{and so} \qquad |\exp z| = 1 \quad \Leftrightarrow \quad \Re z = 0, \tag{11}$$

implying in particular that $\ker(\exp) \subset i\mathbb{R}$.

**Theorem 29.** $\exp : \mathbb{C} \to \mathbb{C}^\times$ *is surjective, and* $\ker(\exp) = iT\mathbb{Z}$ *with some constant* $T > 0$.

*Proof.* Let $A = \exp\mathbb{C}$. By the above lemma, we know that $D_1(1) \subseteq A$. So if $a \in A$, then by the group structure of $A$, it must hold that $az \in A$ for all $z \in D_1(1)$. Since the collection of all such $az$ is just the open disk $D_{|a|}(a)$ centred at $a$, the above amounts to $D_{|a|}(a) \subseteq A$, so that $A$ is open.

Let $b \in B = \mathbb{C}^\times \setminus A$. Then $bA = \{ba : a \in A\}$ is open. Also $bA \cap A = \varnothing$ since otherwise the existence of $c \in bA \cap A$ would imply the existence of $a \in A$ with $c = ba$, which would mean $b = ca^{-1} \in A$. This implies that $B = \cup_{b\in B} bA$ is open, and since $A$ is nonempty, we have $A = \mathbb{C}^\times$.

We shall prove the claim about the kernel $K = \ker(\exp) \subset i\mathbb{R}$. Note the symmetry $K = -K$ since $\exp z = 1$ implies $\exp(-z) = \frac{1}{\exp z} = 1$. By surjectivity, there is $s \in \mathbb{R}$ with $\exp(is) = -1$, and so with $\exp(i2s) = 1$. Obviously $s \neq 0$, and thus $K \neq \{0\}$. Since $K$ contains numbers with positive imaginary parts, the number $T = \inf\{t > 0 : it \in K\}$ is well defined. Moreover, $iT \in K$ because $K$ is closed as the pre-image of $\{1\}$ under a continuous function. So one trivially has $iT\mathbb{Z} \subseteq K$. Since exp is (entire) analytic, the zeroes of $\exp(z)-1$ form a discrete set, which means that there is a neighbourhood $U$ of 0 such that the only solution of $\exp z = 1$ in $U$ is 0. This implies $T > 0$. Now that we know $T > 0$, suppose that $r \in \mathbb{R}$ with $ir \in K$. Then there is $n \in \mathbb{Z}$ such that $nT \leq r < (n+1)T$, or $0 \leq r - nT < T$. But $\exp(ir - inT) = \exp(ir)\exp(-inT) = 1$, hence $r - nT = 0$ by the minimal property of $T$. This proves that $K \subseteq iT\mathbb{Z}$. □

Once we know the kernel and the range, it is easy to study the periods and possible inverses of the exponential function. So $\exp(z+w) = \exp(z)$ is equivalent to $\exp(w) = 1$, which means $w \in iT\mathbb{Z}$. One implication of this is that the periods of the exponential function are precisely the numbers $inT$ with $n \in \mathbb{Z}$. Another, a quite strong implication is that if $I$ is any half-open interval of length $T$, then the function $\exp : \mathbb{R} \times I \to \mathbb{C}^\times$ takes any of the values in $\mathbb{C}^\times$ *precisely once*. Moreover, the surjectivity of $\exp : \mathbb{C} \to \mathbb{C}^\times$ combined with (11) implies that the homomorphism $p : t \mapsto \exp(it)$ sending the reals into the unit circle is surjective with the periods $T\mathbb{Z}$, and similarly to the above, for any half-open interval $I$, the function $p : I \to S^1$ goes through any point on $S^1$ exactly once. Thus every $z \in \mathbb{C}$ with $|z| = 1$ can be written uniquely as $z = p(t)$ with $t \in I$. We have $p'(t) = ip(t)$, and a little more careful analysis (involving integration to find the length of an arc) shows that the central angle between 1 and $p(t)$ counted anti-clockwise from 1 is equal to $t \in I$ when $I$ is taken to be $[0, T)$. This implies by definition the Euler identity

$$\exp(it) = \cos t + i \sin t, \qquad \text{and that} \qquad T = 2\pi. \tag{12}$$

We will also use the short notation $\exp(z) = e^z$ for the exponential function. Thus every $z \in \mathbb{C}^\times$ can be written as $z = |z|e^{i\theta}$ with $\theta \in \mathbb{R}$, and moreover $\theta$ is unique if one requires $\theta \in I$ with any fixed half-open interval $I$ of length $2\pi$. Then with $\zeta = \log|z| + i\theta$, it is clear that $\exp(\zeta) = \exp(\log|z|)\exp(i\theta) = |z|\exp(i\theta) = z$. So for $\rho e^{i\theta} \in \mathbb{C}^\times$ the solutions of $\exp(\zeta) = \rho e^{i\theta}$ are precisely the numbers $\log\rho + i\theta + i2\pi\mathbb{Z}$. We single out the continuous ones among the inverses of the exponential function.

**Definition 30.** A continuous function $\ell : \Omega \to \mathbb{C}$ is called a *logarithm function* on $\Omega$ if $\exp(\ell(z)) = z$ for all $z \in \Omega$.

**Lemma 31.** *Let $\ell \in C(\Omega)$ be a logarithm function on a connected domain $\Omega$. Then any given $\hat{\ell} \in C(\Omega)$ is a logarithm function on $\Omega$ if and only if there is $n \in \mathbb{Z}$ such that $\hat{\ell}(z) = \ell(z) + 2\pi in$ for all $z \in \Omega$.*

*Proof.* We immediately have $\exp(\hat{\ell}(z) - \ell(z)) = 1$, or $\hat{\ell}(z) - \ell(z) \in 2\pi i\mathbb{Z}$ for all $z \in \Omega$. Since $\Omega$ is connected and $\hat{\ell} - \ell$ is continuous, the image of $\hat{\ell} - \ell$ must be a single point in $2\pi i\mathbb{Z}$. □

*Example* 32. The function $\lambda$ defined in (10) is a logarithm function on $D_1(1)$. Moreover, if $a \in \mathbb{C}^\times$ and $\exp\alpha = a$, then $\lambda_a(z) = \lambda(\frac{z}{a}) + \alpha$ is a logarithm function on $D_{|a|}(a)$.

Another example is the so-called *principal branch*, defined on $\mathbb{C}^- = \mathbb{C}\setminus(-\infty, 0]$ as $\mathrm{Log}(\rho e^{i\theta}) = \log\rho + i\theta$ with $\theta \in (-\pi, \pi)$.

We have $\lambda_a \in C^\omega(D_{|a|}(a))$, and from the above lemma any logarithm function on a neighbourhood of $a$ must be equal to $\lambda_a$ up to an additive constant. In particular it must be analytic on the neighbourhood.

**Corollary 33.** *If $\ell \in C(\Omega)$ is a logarithm function on $\Omega$, then $\ell \in C^\omega(\Omega)$.*

**Definition 34.** Let $\ell : \Omega \to \mathbb{C}$ be a logarithm function on $\Omega$. Then for $\sigma \in \mathbb{C}$, the function $p_\sigma(z) = \exp(\sigma\ell(z))$ is called a *power function* on $\Omega$.

We can derive right away that power functions satisfy $p'_\sigma = \sigma p_{\sigma-1}$, $p_{\sigma+\tau} = p_\sigma p_\tau$, and $p_n(z) = z^n$ if $n \in \mathbb{N}$.

The notation $z^\sigma$ usually stands for $\exp(\sigma \mathrm{Log}\, z)$ involving the principal branch.