MATH 566 LECTURE NOTES 1: HARMONIC FUNCTIONS

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1. The mean value property

In this set of notes, we consider *real-valued* functions on two-dimensional domains, although it is straightforward to generalize the results to, e.g., vector-valued functions defined on n-dimensional domains. The symbol $D_r(z)$ denotes the open disk with radius r and centre z, and we set $D_r = D_r(0)$. Unless otherwise specified, in what follows Ω will be an open subset of $\mathbb{R}^2$.

Definition 1. Let $\Omega \subseteq \mathbb{R}^2$ be open. Then a continuous function $u \in C(\Omega)$ is said to have the *mean value property* and written $u \in \mathrm{MVP}(\Omega)$, if it satisfies

$$u(z) = \frac{1}{\pi r^2} \int_{D_r(z)} u(w)dw, \qquad \text{for all} \quad D_r(z) \subset \Omega.$$

It is easy to show (by differentiating and integrating with respect to r) that the above condition is equivalent to

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r \operatorname{cis} \theta)d\theta, \qquad \text{for all} \quad D_r(z) \subset \Omega,$$

where we have used the shorthand notation $\operatorname{cis}\theta = (\cos\theta, \sin\theta) \in \mathbb{R}^2$. Let us also introduce the notation $\mathrm{Har}(\Omega) = \{u \in C^2(\Omega) : \Delta u = 0\}$ for the set of *harmonic functions* in Ω.

Theorem 2. $\mathrm{Har}(\Omega) \subseteq \mathrm{MVP}(\Omega)$.

Proof. Let $u \in \mathrm{Har}(\Omega)$, and let $D_r(z) \subset \Omega$ and $\rho \in (0, r)$. Then obviously the following integral must vanish:

$$\int_{D_\rho(z)} \Delta u = \int_{\partial D_\rho(z)} \partial_\nu u dl = \rho \int_0^{2\pi} \partial_\rho u(z + \rho \operatorname{cis}\theta)d\theta = \rho\partial_\rho \int_0^{2\pi} u(z + \rho \operatorname{cis}\theta)d\theta,$$

where commuting the derivative ∂_ρ with the integration over θ is justified because the functions $(\rho, \theta) \mapsto u(z + \rho \operatorname{cis}\theta)$ and $(\rho, \theta) \mapsto \partial_\rho u(z + \rho \operatorname{cis}\theta)$ are both continuous. Now from the fundamental theorem of calculus we have

$$\partial_\rho \int_0^{2\pi} u(z + \rho \operatorname{cis}\theta)d\theta = 0 \qquad \Rightarrow \qquad \int_0^{2\pi} u(z + r \operatorname{cis}\theta)d\theta - 2\pi u(z) = 0.$$

□

So viewed as a tool, the mean value property can be used to prove properties of harmonic functions. The following converse shows that the mean value property can also be used to prove harmonicity.

Theorem 3. $\mathrm{MVP}(\Omega) \subseteq \mathrm{Har}(\Omega)$ *and* $\mathrm{MVP}(\Omega) \subseteq C^\infty(\Omega)$.

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Proof. Let $\varphi \in C_0^\infty(D_1)$ be a positive function with $\int \varphi = 1$, which is also radial, i.e., $\varphi(z) = \psi(|z|)$ for some ψ. The normalization of φ implies $2\pi \int_0^1 r\psi(r)dr = 1$. Set $\varphi_\varepsilon(z) = \varepsilon^{-2}\varphi(z/\varepsilon)$ for $\varepsilon > 0$, and assume that $z \in \Omega$ and $\varepsilon < \mathrm{dist}(z, \partial\Omega)$. Then

$$\int_\Omega u(w)\varphi_\varepsilon(w-z)dw = \int_\Omega u(z+w)\varphi_\varepsilon(w)dw = \frac{1}{\varepsilon^2}\int_{|w|<\varepsilon} u(z+w)\psi(\frac{|w|}{\varepsilon})dw$$
$$= \frac{1}{\varepsilon^2}\int_{r<\varepsilon}\int_0^{2\pi} u(z+r\,\mathrm{cis}\,\theta)\psi(\frac{r}{\varepsilon})rdrd\theta = \frac{2\pi u(z)}{\varepsilon^2}\int_{r<\varepsilon}\psi(\frac{r}{\varepsilon})rdr = u(z),$$

which means that $u = \varphi_\varepsilon * u$, and so $u \in C^\infty(\Omega)$.

Now let $D_r(z) \subset \Omega$, and compute

$$\int_{D_r(z)} \Delta u(w)dw = r\partial_r \int_0^{2\pi} u(z+r\,\mathrm{cis}\,\theta)d\theta = r\partial_r(2\pi u(z)) = 0.$$

Since $D_r(z)$ is arbitrary, we conclude that $\Delta u = 0$ in Ω. □

2. Simple consequences of the mean value property

Recall that an open set Ω is *connected* if it cannot be decomposed as the disjoint union of two open sets.

Theorem 4 (Maximum principle)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open, and let $u \in C^2(\Omega)$ be a subharmonic function in Ω, i.e., $\Delta u \geq 0$ in Ω.*

(a) If Ω is connected and $u(z) = \sup_\Omega u$ at some $z \in \Omega$, then u is constant.

(b) If Ω is bounded and $u \in C(\bar\Omega)$, then we have $\sup_\Omega u \leq \max_{\partial\Omega} u$.

Proof. For part (a), let $M = u(z) = \sup_\Omega u$. Then we can write the mean value property as

$$\int_0^{2\pi} [u(z) - u(z+r\,\mathrm{cis}\,\theta)]d\theta = 0, \qquad \text{for all sufficiently small } r > 0.$$

The integrand is a nonnegative continuous function, so it must vanish, which implies that $u(w) = M$ in a small disk $w \in D_r(z)$. In other words, the set $\Sigma = \{z \in \Omega : u(z) = M\}$ is open. Since u is continuous, the set $\{z \in \Omega : u(z) < M\} = \Omega \setminus \Sigma$ is also open. This means that either $\Sigma = \Omega$ or $\Sigma = \varnothing$, but Σ is not empty by hypothesis.

For part (b), there is $z \in \bar\Omega$ with $u(z) = \sup_{\bar\Omega} u$ since Ω is bounded and u is continuous on $\bar\Omega$. If $z \in \partial\Omega$ then we are done; otherwise applying part (a) to the connected component of Ω that contains z concludes the proof. □

Remark 5. In part (b) of the above theorem, the condition that Ω be bounded cannot be simply removed, as can be seen from the example $u(x, y) = e^x \cos y$ on $(0, \infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$.

In the following theorem we use complex notation for convenience.

Theorem 6 (Schwarz reflection principle)**.** *Let $\Omega \subset \mathbb{C}$ be an open set symmetric with respect to the real axis, and let $\Omega^+ = \Omega \cap \{\Im z > 0\}$ be the part of it in the upper half plane. Moreover, assume that $u \in \mathrm{Har}(\Omega)$, and that $u(z) \to 0$ as $z \in \Omega^+$ tends to any point on the real axis $\{\Im z = 0\}$. Then u extends to be harmonic on Ω, and the extension satisfies*

$$u(\bar z) = -u(z), \qquad z \in \Omega.$$

Proof. Let $\Omega^- = \{\bar{z} : z \in \Omega^+\}$, and extend u to a function on Ω by $u(z) = 0$ for $z \in \Omega \cap \{\Im z = 0\}$ and $u(z) = -u(\bar{z})$ for $z \in \Omega^-$. Then the mean value property is obviously satisfied at points $z \in \Omega^+ \cup \Omega^-$. But for (the remaining) points z with $\Im z = 0$, since $u(z) = -u(\bar{z})$, the integral of u over any small circle centred at z is zero, giving the mean value property for those points. $\square$

For $\Omega \subseteq \mathbb{R}^2$ and a function $u \in C(\Omega)$, we define the *uniform norm*

$$\|u\|_\Omega = \sup_\Omega |u|.$$

We say that a sequence of functions $\{u_k\} \subset C(\Omega)$ *converges uniformly* on Ω to $u \in C(\Omega)$, if $\|u_k - u\|_\Omega \to 0$. Recall that continuity and the Riemann integral are preserved under uniform limits.

Theorem 7. *If $\{u_k\} \subset \mathrm{Har}(\Omega)$ converges uniformly on Ω to u, then $u \in \mathrm{Har}(\Omega)$.*

Proof. We have $u \in C(\Omega)$. Let $D_r(z) \subset \Omega$. Then we have

$$2\pi u_k(z) = \int_0^{2\pi} u_k(z + r \operatorname{cis}\theta) d\theta, \qquad \text{for all } k.$$

Since the left hand side converges to $2\pi u(z)$, and the integrand converges uniformly to $\theta \mapsto u(z + r\operatorname{cis}\theta)$, the mean value property over $\partial D_r(z)$ is satisfied for u, and since $D_r(z)$ is arbitrary we have $u \in \mathrm{Har}(\Omega)$. $\square$

3. Derivative bounds

Noting that partial derivatives of harmonic functions are also harmonic, and by using the mean value property for the partial derivatives, we can bound the derivatives of harmonic functions by the size of the function itself. Recall that for $\nu = (\nu_1, \nu_2)$ with $|\nu| = 1$, the *directional derivative* along ν is defined by $\partial_\nu u = \nu_1 \partial_x u + \nu_2 \partial_y u$.

Theorem 8. *Let $u \in \mathrm{Har}(\Omega)$ and $\bar{D}_r(z) \subset \Omega$. Then for any $\nu \in \mathbb{R}^2$ with $|\nu| = 1$, we have*

$$|\partial_\nu u(z)| \leq \frac{2}{r} \max_{\partial D_r(z)} |u|.$$

If in addition $u \geq 0$ in $\bar{D}_r(z)$, we have

$$|\partial_\nu u(z)| \leq \frac{2}{r} u(z).$$

Proof. Since $\Delta u = 0$ and u is smooth, we have $\Delta \partial_\nu u = 0$ in Ω, i.e., $\partial_\nu u \in \mathrm{Har}(\Omega)$. Using the mean value property and the divergence theorem, we get

$$\partial_\nu u(z) = \frac{1}{\pi r^2} \int_{D_r(z)} \partial_\nu u(w) dw = \frac{1}{\pi r^2} \int_0^{2\pi} u(z + r\operatorname{cis}\theta)(\nu \cdot \operatorname{cis}\theta) r d\theta.$$

Now if $u \geq 0$, since the inner product $|\nu \cdot \operatorname{cis}\theta| \leq 1$, we infer

$$\partial_\nu u(z) \leq \frac{1}{\pi r} \int_0^{2\pi} u(z + r\operatorname{cis}\theta) d\theta = \frac{1}{\pi r} \cdot 2\pi u(z).$$

On the other hand, for general u, we have

$$|\partial_x u(z)| \leq \frac{1}{\pi r} \max_{\partial D_r(z)} |u| \cdot 2\pi = \frac{2}{r} \max_{\partial D_r(z)} |u|.$$

$\square$

The following can be called the Liouville theorem for harmonic functions.

Corollary 9. *If $u \in \mathrm{Har}(\mathbb{R}^2)$ is bounded above or bounded below, then u is constant.*

Proof. If $u \geq a$ for some constant $a \in \mathbb{R}$, then set $v = u - a$, or if $u \leq b$ for some constant $b \in \mathbb{R}$, then set $v = b - u$. In either case, we have $v \geq 0$ and $v \in \mathrm{Har}(\mathbb{R}^2)$. Now let $z \in \mathbb{R}^2$ and apply the above theorem to v, inferring

$$|\partial_\nu v(z)| \leq \frac{2}{r} v(z), \qquad \text{for any} \quad r > 0,$$

which gives $\partial_\nu v \equiv 0$ in any direction ν. □

Now we estimate higher derivatives.

Corollary 10. *Let $u \in \mathrm{Har}(\Omega)$ and $\bar{D}_r(z) \subset \Omega$. Then for $j, k \geq 0$ and $n = j + k$, we have*

$$|\partial_x^j \partial_y^k u(z)| \leq \frac{2^n n^n}{r^n} \max_{\bar{D}_r(z)} |u|.$$

Proof. We prove it only for $k = 0$, since the general case is completely analogous. All the derivatives $\partial_x^m u$ are harmonic, so with $\rho = r/n$, we have

$$|\partial_x^n u(z)| \leq \frac{2}{\rho} \max_{\bar{D}_\rho(z)} |\partial_x^{n-1} u| \leq \ldots \leq \left(\frac{2}{\rho}\right)^n \max_{\bar{D}_{n\rho}(z)} |u|,$$

which is what we need. □

4. Analyticity

Let us recall the Taylor theorem for two variables, with a somewhat simplified hypothesis on the smoothness of the function involved.

Theorem 11. *Let $f \in C^n(\bar{D}_r(z))$, and $h = (h_1, h_2) \in D_r(0)$. Then we have*

$$f(z+h) = \sum_{j+k<n} \frac{\partial_x^j \partial_y^k f(z)}{j!k!} h_1^j h_2^k + \sum_{j+k=n} R_{j,k}(z) h_1^j h_2^k,$$

with the following estimate on the error term

$$|R_{j,k}(z)| \leq \sup_{w \in \bar{D}_r(z)} \left| \frac{\partial_x^j \partial_y^k f(w)}{j!k!} \right|.$$

Definition 12. A function $f \in C^\infty(\Omega)$ is called *(real) analytic at* $z \in \Omega$ if the following Taylor series converges in a neighbourhood of z.

$$f(z+h) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\partial_x^j \partial_y^k f(z)}{j!k!} h_1^j h_2^k.$$

Moreover, f is said to be *(real) analytic in* Ω if it is analytic at each $z \in \Omega$. The set of analytic functions in Ω is denoted by $C^\omega(\Omega)$.

Theorem 13. $\mathrm{Har}(\Omega) \subseteq C^\omega(\Omega)$.

Proof. Let $D_r(z) \subset \Omega$ and $\rho \in (0, r)$. Then we have

$$|R_{j,k}(z)| \leq \sup_{\bar{D}_\rho(z)} \left| \frac{\partial_x^j \partial_y^k u}{j!k!} \right| \leq \frac{2^n n^n}{j!k!(r-\rho)^n} \max_{\bar{D}_r(z)} |u|$$

Using the crude bounds $\binom{n}{j} \leq 2^n$ and $n^n \leq e^n n!$, we get

$$\frac{2^n n^n}{j!k!} = \binom{n}{j} \frac{2^n n^n}{n!} \leq 4^n e^n,$$

and with this in mind, for $|h| \leq \rho$ the Taylor remainder term can be estimated as

$$|R_{j,k}(z)h_1^j h_2^k| \leq \left(\frac{4e\rho}{r-\rho}\right)^n \max_{\bar{D}_r(z)} |u|.$$

We see that the Taylor series converges in $D_\rho(z)$ whenever $\rho < \frac{r}{4e+1}$. □

As a consequence of analyticity, we have the following rigidity that is typical in analytic setting. What is remarkable is that there is no condition on the size of D relative to Ω; we only need D nonempty and open.

Theorem 14 (Identity theorem)**.** *Let $u \in \mathrm{Har}(\Omega)$ with Ω a connected open set, and let $u \equiv 0$ in an open subset $D \subset \Omega$. Then $u \equiv 0$ in Ω.*

Proof. Each $\Sigma_{j,k} = \{z \in \Omega : \partial_x^j \partial_y^k u(z) = 0\}$ is closed, so the intersection $\Sigma = \bigcap_{j,k} \Sigma_{j,k}$ is also closed. But Σ is also open, because $z \in \Sigma$ implies that $u \equiv 0$ in a small disk centred at z by a Taylor series argument. Since Σ is nonempty by hypothesis, we conclude that $\Sigma = \Omega$. □

The following corollary records the fact that a harmonic function is completely determined by its restriction to any open subset of its domain of definition. In other words, if it is at all possible to extend a harmonic function (defined on an open set) to a bigger domain, then there is only one way to do the extension.

Corollary 15. *Let $u, v \in \mathrm{Har}(\Omega)$ with Ω a connected open set, and let $u \equiv v$ in an open subset $D \subset \Omega$. Then $u \equiv v$ in Ω.*