MATH 387 PRACTICE PROBLEMS

WINTER 2018

1. Analyze the convergence of the fixed point iteration

$$x_{n+1} = x_n + \kappa \sin x_n,$$

for computing the solutions of $\sin(x) = 0$, where $\kappa \neq 0$ is a constant. That is, how do the existence as well as the value of the limit $\lim x_n$ depend on the initial guess x_0, and what is the order of convergence? Of course, the answers will most likely depend on the value of κ. Sketch a cobweb plot of the iterations.

2. Show that the equation $e^x = x + 2$ has two real solutions, $\alpha < 0$ and $\beta > 0$. Letting $x_0, x_1, \ldots$ denote the iterates of the Newton-Raphson method applied to this equation, show that if $x_0 < 0$ then $x_n \to \alpha$ as $n \to \infty$, and if $x_0 > 0$ then $x_n \to \beta$ as $n \to \infty$.
3. Let $A \in \mathbb{R}^{n\times n}$ be a tridiagonal matrix which admits an LU decompoistion, and let $\tilde{L}$ and $\tilde{U}$ be the result of a computation of the LU decomposition in floating point arithmetic. Estimate the entries of $A - \tilde{L}\tilde{U}$, and argue that this process is backward stable.
4. Show that the minimax polynomial approximation of $f \in C([-a, a])$ of any degree is an even function, if f is even. Deduce that if f is even, its minimax polynomial of degree $2n$ is also the minimax polynomial of degree $2n + 1$.
5. In each of the following cases, find the minimax polynomial approximation of degree n for the function $f(x)$ on the interval $[a, b]$. You need to prove that the polynomial you found is indeed the minimax polynomial.
   (a) $f(x) = \sin x$, $[a, b] = [-1, 1]$, $n = 2$.
   (b) $f(x) = \cos x^2$, $[a, b] = [-1, 1]$, $n = 3$.
   (c) $f(x) = |x|$, $[a, b] = [-1, 2]$, $n = 1$.
6. Construct orthogonal polynomials of degrees 0, 1, and 2 on the interval $(0, 1)$ with respect to the weight function
   (a) $w(x) = \log \frac{1}{x}$.
   (b) $w(x) = \frac{1}{\sqrt{x}}$.
7. In each of the following cases, find the least squares approximation polynomials of degrees 0, 1 and 2 for the function $f(x)$ on the interval (a, b) with respect to the weight function $w(x) \equiv 1$.
   (a) $f(x) = \sin x$, $(a, b) = (-\pi, \pi)$.
   (b) $f(x) = \sin x$, $(a, b) = (-\frac{\pi}{2}, \frac{\pi}{2})$.
   (c) $f(x) = \sin x$, $(a, b) = (0, \pi)$.

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8. Compute weights and nodes of the quadrature formula

$$\int_0^1 f(x)w(x)\,\mathrm{d}x \approx \omega_0 f(x_0) + \omega_1 f(x_1),$$

so that the order of the quadrature is maximum, where the weight function is
(a) $w(x) = \log \frac{1}{x}$.
(b) $w(x) = \frac{1}{\sqrt{x}}$.

9. In each of the following cases, compute weights and nodes of the quadrature formula

$$\int_a^b w(x)f(x)\,\mathrm{d}x \approx \omega_0 f(x_0) + \omega_1 f(x_1) + \ldots + \omega_n f(x_n),$$

so that the order (or equivalently, the degree of exactness) of the quadrature is maximum.
(a) $w(x) = 2x^2 + 1$, $(a, b) = (-1, 1)$, $n = 0$.
(b) $w(x) = 1 + \theta(x)$, $(a, b) = (-1, 1)$, $n = 1$, where θ is the Heaviside step function.
(c) $w(x) = \sin x$, $(a, b) = (0, \frac{\pi}{2})$, $n = 1$.
(d) $w(x) = e^{-x}$, $(a, b) = (0, \infty)$, $n = 3$.