MATH 387 TAKE HOME MIDTERM EXAM

DUE SATURDAY MARCH 17, 18:00 EST

Note: Submit your work as a Jupyter notebook. The rules are the same as in the previous assignments, except that no collaboration is now allowed. The subject line of your email containing the GitHub link should be: `[Math 387 Midterm] Yourname`

Part I. Theory

(Refrain from using additional reading material, and attempt to finish it in 30 min)

Let f be a C^3 function on some interval $[-a, a]$, and consider the first order centred difference formula

$$D_h = \frac{f(h) - f(-h)}{2h},$$

for approximating $f'(0)$, where $h > 0$ is assumed to be small. In exact arithmetic, it is easy to see that $D_h - f'(0) = O(h^2)$. Thus the accuracy of D_h can be made arbitrarily small by choosing the parameter h small. Now, suppose that the function evaluations are performed inexactly, so that instead of f, we only have access to some function $f + \delta f$ with $|\delta f(x)| \leq \varepsilon$ for $x \in (-a, a)$, where $\varepsilon > 0$ is a small constant. The aforementioned centred difference should then be replaced by

$$\tilde{D}_h = \frac{f(h) + \delta f(h) - f(-h) - \delta f(-h)}{2h}.$$

(a) Show that for each $0 < h < a$, there exists $\xi \in (-h, h)$ such that

$$\tilde{D}_h - f'(0) = \frac{h^2}{6} f'''(\xi) + \frac{\delta f(h) - \delta f(-h)}{2h}.$$

(b) Conclude that

$$|\tilde{D}_h - f'(0)| \leq \frac{\varepsilon}{h} + \frac{Mh^2}{6}, \qquad \text{where} \qquad M = \max_{x \in [-a,a]} |f'''(x)|.$$

Supposing that ε and M are fixed, sketch the graph of the bound $B(h) = \frac{\varepsilon}{h} + \frac{Mh^2}{6}$ as a function of h. Find the value of h that minimizes $B(h)$. Explain why it is not a good idea to choose h too small.

Date: Winter 2018.

Part II. Lab

(You are allowed to use additional reading material)

It is well known that Lagrange interpolation with equally spaced nodes is susceptible to the so-called *Runge phenomenon*. In this exercise, you are asked to do some reading to familiarize yourself with this phenomenon, and illustrate the phenomenon by a well-chosen numerical experiment. The design of the experiment, and how you showcase it are entirely up to you, with the only requirement being that you need to show some originality. In particular, the standard example $f(x) = \frac{1}{1+x^2}$ and its close relatives should not be used. You are *not* required to provide a rigorous mathematical proof that your example indeed exhibits the Runge phenomenon.

(a) Perform Lagrange interpolation with equally spaced nodes, to illustrate the Runge phenomenon.

(b) Perform Lagrange interpolation with Chebyshev nodes on the same examples, in order to see how Chebyshev nodes compare to equally spaced nodes. You need to find out what Chebyshev nodes are, in case you are not familiar with them.

(c) Compute the Bernstein polynomials (obviously with equally spaced nodes) on the same examples, to see how this approach compares to interpolation.