MATH 387 ASSIGNMENT 4

DUE WEDNESDAY APRIL 11

Note: As usual, you are encouraged to do additional reading, and strongly encouraged to type your solutions in LaTeX.

1. The outcome of this exercise will be used in Lab Assignment 3.
   (a) Show that
   $$\|L_n\| = \max_{x\in[a,b]} \sum_{k=0}^{n} |\phi_{n,k}(x)|, \tag{1}$$
   where $\phi_{n,k}$ is the k-th Lagrange basis function associated to the nodes $x_0, \ldots, x_n$, and $\|L_n\|$ is the Lebesgue constant of the corresponding Lagrange interpolation.
   (b) Derive an expression of the form
   $$\|S_n\| = \int_{-1}^{1} \Big| \sum_{k=0}^{n} a_k P_k(x) \Big| dx, \tag{2}$$
   for the Lebesgue constants of the Legendre truncation, where P_k are the Legendre polynomials.
   (c) For interpolatory quadrature with the nodes $x_0, \ldots, x_n$, for approximating the integral over (a, b) with weight $w(x) = 1$, show that
   $$\|Q_n\| = \sum_{k=0}^{n} |\omega_k| = \sum_{k=0}^{n} \Big| \int_a^b \phi_{n,k}(x) dx \Big|, \tag{3}$$
   where $\phi_{n,k}$ is the k-th Lagrange basis function associated to the given nodes, and $\|Q_n\|$ is the "Lebesgue constant" of the quadrature formula, cf. Lab Assignment 3.
2. Recall that the Bézier curve associated to the control points $P_0, \ldots, P_n \in \mathbb{R}^2$ is given by
   $$B(t) = \sum_{k=0}^{n} \beta_{n,k}(t) P_k, \qquad 0 \le t \le 1,$$
   where
   $$\beta_{n,k}(t) = \binom{n}{k} (1-t)^{n-k} t^k,$$
   are the Bernstein basis polynomials. Show that the velocity of the Bézier curve satisfies
   $$B'(t) = n \sum_{k=0}^{n-1} \beta_{n-1,k}(t)(P_{k+1} - P_k), \qquad 0 \le t \le 1.$$

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3. (a) Find the minimax polynomial approximation of degree n for the function
$$f(x) = a_0 + a_1 x + \ldots + a_{n+1} x^{n+1}$$
on the interval $[-1, 1]$, where $a_{n+1} \neq 0$. (You need to prove that the polynomial you found is indeed the minimax polynomial.)
   (b) Find the least squares approximation polynomials of degrees $n = 0, 1, 2, 3, 4$ for $f(x) = |x|$ on the interval $(-1, 1)$ with respect to the weight function $w(x) \equiv 1$.
4. (Hermite/osculatory interpolation) Let $x_0, x_1, \ldots, x_n$ be distinct points in $[a, b]$. Show that if f and its first derivative are defined respectively at the points $x_0, x_1, \ldots, x_n$, then there exists a unique polynomial q of degree at most $2n + 1$, such that
$$q(x_j) = f(x_j), \quad \text{and} \quad q'(x_j) = f'(x_j), \qquad \text{for} \quad j = 0, 1, \ldots, n.$$
Furthermore, prove that if $f \in C^{2n+2}([a, b])$, then for any $x \in [a, b]$, there exists $\xi \in [a, b]$ such that
$$f(x) - q(x) = \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n)!} f^{(2n+2)}(\xi).$$
5. For functions $f \in C([a, b])$ where $-\infty < a < b < \infty$, and for $1 < p < \infty$, define the *p-norm*
$$\|f\|_p = \Big(\int_a^b |f(x)|^p \mathrm{d}x \Big)^{\frac{1}{p}},$$
and consider the problem of approximating f by polynomials in the p-norm: Find $q \in \mathbb{P}_n$ such that $\|f - q\|_p$ is minimal.
   (a) Show that for any $f \in C([a, b])$, there exists $g_n \in \mathbb{P}_n$ such that
$$\|f - g_n\|_p = \inf_{q \in \mathbb{P}_n} \|f - q\|_p.$$
   (b) Show that the best approximation $g_n \in \mathbb{P}_n$ as in (a) is unique.
   (c) Show that g_n converges to f in the p-norm as $n \to \infty$.
   (d) Design an algorithm to compute g_n.