MATH 387 ASSIGNMENT 2

DUE FRIDAY FEBRUARY 23

Note: You are encouraged to type your solutions in LaTeX.

1. In each of the following cases, analyze the convergence of the fixed point iteration

$$x_{n+1} = \phi(x_n),$$

for computing the solutions of $f(x) = 0$. That is, how do the existence as well as the value of the limit $\lim x_n$ depend on the initial guess x_0, and what is the order of convergence? Sketch a cobweb plot of the iteration.
   (a) $\phi(x) = x^2 - 2$, $f(x) = x^2 - x - 2$.
   (b) $\phi(x) = -\sqrt{x+2}$, $f(x) = x^2 - x - 2$.
   (c) $\phi(x) = x - 2 + \frac{x}{x-1}$, $f(x) = \frac{2x^2-3x-2}{x-1}$.
2. In each of the following cases, propose two different fixed point methods for approximating the root $x = \alpha$ of $f(x) = 0$, such that one method is linearly convergent, and the other is quadratically convergent. Give detailed proofs of convergence.
   (a) $f(x) = e^{-x} - \sin x$, and α is the smallest positive root.
   (b) $f(x) = x - \cos x$, and α is the smallest positive root.
3. (Trefethen-Bau) Recall that Gaussian elimination yields a factorization $A = LU$, where L has unit diagonal but U in general does not. Describe the factorization that results if this process is varied in the following ways.
   (a) Elimination by columns from left to right, rather than by rows from top to bottom, so that A is made lower triangular.
   (b) Gaussian elimination applied after a preliminary scaling of the columns of A by a diagonal matrix D. What form does a system $Ax = b$ take under this rescaling? Is it the equations or the unknowns that are rescaled by D?
   (c) Gaussian elimination carried further, so that after A (assumed nonsingular) is brought to upper triangular form, additional operations ("backward elimination") are carried out so that this upper triangular matrix is made diagonal.
4. (Trefethen-Bau) Gaussian elimination $PA = LU$ can be used to compute the inverse A^{-1} of a nonsingular matrix $A \in \mathbb{R}^{n\times n}$, although it is rarely really necessary to do so.
   (a) Describe an algorithm for computing A^{-1} by solving n systems of equations, and show that the number of floating point multiplication/division operations taken by the algorithm is bounded by $Cn^3 + O(n^2)$ as $n \to \infty$. What is the best value for C?
   (b) Describe a variant of your algorithm, taking advantage of sparsity, that reduced the operation count to $cn^3 + O(n^2)$ with $c \sim C/2$.

Date: Winter 2018.

(c) Suppose one wishes to solve m systems of equations $Ax^{(k)} = b^{(k)}$, $k = 1, \dots, m$, or equivalently, a block system $AX = B$ with $B \in \mathbb{R}^{n \times m}$. What is the asymptotic operation count (a function of n and m) for doing this (i) directly from the LU factorization, and (ii) with a preliminary computation of A^{-1}?

5. Let $A \in \mathbb{R}^{m \times n}$, and consider the matrix-vector multiplication problem $(A, x) \mapsto y = Ax$, where $x \in \mathbb{R}^n$. Let $\tilde{y} \in \mathbb{R}^n$ be the result of a computation of Ax in floating point arithmetic (with the "machine epsilon" $\varepsilon > 0$). Show that there exists a matrix $\tilde{A}$, such that $\tilde{A}x = \tilde{y}$ in exact arithmetic and that the entries of $\tilde{A} - A$ can be bounded in absolute value by an expression depending only on ε, n, m, and A. Argue that matrix-vector multiplication is backward stable. Then estimate the error $\|\tilde{y} - y\|$.