SAMPLE SOLUTIONS FOR ASSIGNMENT 1

MATH 387 WINTER 2016

Problem 5c

Suppose that x_0 is an initial approximation to the exact inverse $x = \frac{1}{a}$, and let us try to find h such that $x_0 + h$ is a better approximation. Let $\Delta = a - \frac{1}{x_0}$, which we think of as small. We want

$$(x_0 + h)a = (x_0 + h)(\frac{1}{x_0} + \Delta) = 1,$$

that is,

$$1 + x_0\Delta + \frac{h}{x_0} + h\Delta = 1.$$

Ignoring the quadratically small term $h\Delta$, we get

$$h = -x_0^2\Delta = x_0 - ax_0^2, \qquad \text{i.e.,} \qquad x + h = 2x_0 - ax_0^2.$$

This leads to the following iteration

$$x_{n+1} = 2x_n - ax_n^2,$$

involving only multiplication and subtraction.

As a consistency check, we have $2x - ax^2 = x$ for $x = \frac{1}{a}$, so the exact inverse is a fixed point of our iteration. To study its convergence, let $e_n = x_n - x$, so that $x_n = x + e_n$, and derive

$$\begin{aligned} x + e_{n+1} &= 2(x + e_n) - a(x + e_n)^2 = 2x + 2e_n - ax^2 - 2axe_n - ae_n^2 \\ &= x - ae_n^2, \end{aligned}$$

where we have used $ax = 1$. Thus we have

$$e_{n+1} = -ae_n^2.$$

In terms of the relative error $\varepsilon_n = e_n/x = ae_n$, defined without the absolute value for convenience, we have

$$\varepsilon_{n+1} = -\varepsilon_n^2,$$

meaning that the iteration is quadratically convergent. This also shows that the iteration is convergent if and only if $|\varepsilon_0| < 1$. The latter condition is equivalent to $-1 < a(x_0 - x) < 1$, that is, $0 < ax_0 < 2$. In particular, the iteration is not globally convergent, but it is not difficult to pick an initial guess x_0 that ensures convergence.

Supposing that $|\varepsilon_0| < 1$, we have

$$|\varepsilon_n| = |\varepsilon_{n-1}|^2 = \ldots = |\varepsilon_0|^{2^n},$$

which is an *a priori* error estimator (This is a very special situation where we actually get an exact equality). As for an *a posteriori* error estimator, we start with the idea that since $x_na - 1 = 0$ would mean that x_n is the exact solution (hence the error is 0), the difference $x_na - 1$ must have some information on the error (this quantity is called the *residual*). When we follow this thread a bit, we get a nice surprise

$$x_na - 1 = x_na - xa = e_na = \varepsilon_n,$$

and so the quantity

$$\eta_n = |x_n a - 1|,$$

can be used as an *a posteriori* error estimator. In this special situation we have $\eta_n = |\varepsilon_n|$, which means that η_n is not just an estimator of the error, it *is* the actual error.