SOLUTIONS TO PRACTICE PROBLEMS 1, 4, 5(A), 6(B), AND 8(D)

MATH 387 WINTER 2016

Problem 1

Analyze the convergence of the fixed point iteration

$$x_{n+1} = x_n + \kappa \sin x_n,$$

for computing the solutions of $\sin(x) = 0$, where $\kappa \neq 0$ is a constant. That is, how do the existence as well as the value of the limit $\lim x_n$ depend on the initial guess x_0, and what is the order of convergence? Of course, the answers will most likely depend on the value of κ. Sketch a cobweb plot of the iterations.

Solution

Let $\phi(x) = x + \kappa \sin x$. If $x_n \to \alpha$ for some α, then $x_{n+1} = \phi(x_n) \to \phi(\alpha)$ by continuity, meaning that $x_n \to \phi(\alpha)$. Hence we conclude that $\alpha = \phi(\alpha)$, that is, α must be a fixed point of ϕ. The fixed points of ϕ are easily found to be

$$\alpha = \pi m, \qquad m \in \mathbb{Z}.$$

We know that the local behaviour of the iteration is dictated by the derivatives

$$\phi'(\pi m) = 1 + (-1)^m \kappa.$$

For m even, $\alpha = \pi m$ is a stable fixed point when $-2 < \kappa < 0$, and unstable when $\kappa < -2$ or $\kappa > 0$. For m odd, $\alpha = \pi m$ is a stable fixed point when $0 < \kappa < 2$, and unstable when $\kappa < 0$ or $\kappa > 2$. This information is better displayed as a table:

	$\kappa < -2$	$-2 < \kappa < 0$	$0 < \kappa < 2$	$\kappa > 2$
m even	unstable	stable	unstable	unstable
m odd	unstable	unstable	stable	unstable

Case $\kappa < -2$ or $\kappa > 2$. To clarify what we mean by stable and unstable fixed points, let α be a fixed point, and let $x \approx \alpha$. Then we have

$$\phi(x) - \alpha = \phi'(\alpha)(x - \alpha) + O(|x - \alpha|^2), \tag{1}$$

and so if $|x - \alpha|$ is sufficiently small, then $|\phi'(\alpha)| > 1$ implies $|\phi(x) - \alpha| > |x - \alpha|$, and $|\phi'(\alpha)| < 1$ implies $|\phi(x) - \alpha| < |x - \alpha|$. This means that when $\kappa < -2$ or $\kappa > 2$, one cannot have the convergence $x_n \to \alpha$ with $x_n \neq \alpha$, cf. Figure 1. The only possibility of convergence in this case is if $x_n = \alpha$ for some finite n. However, in order to have $x_n = \alpha$, the initial condition x_0 must be carefully chosen (In fact there are only countably many

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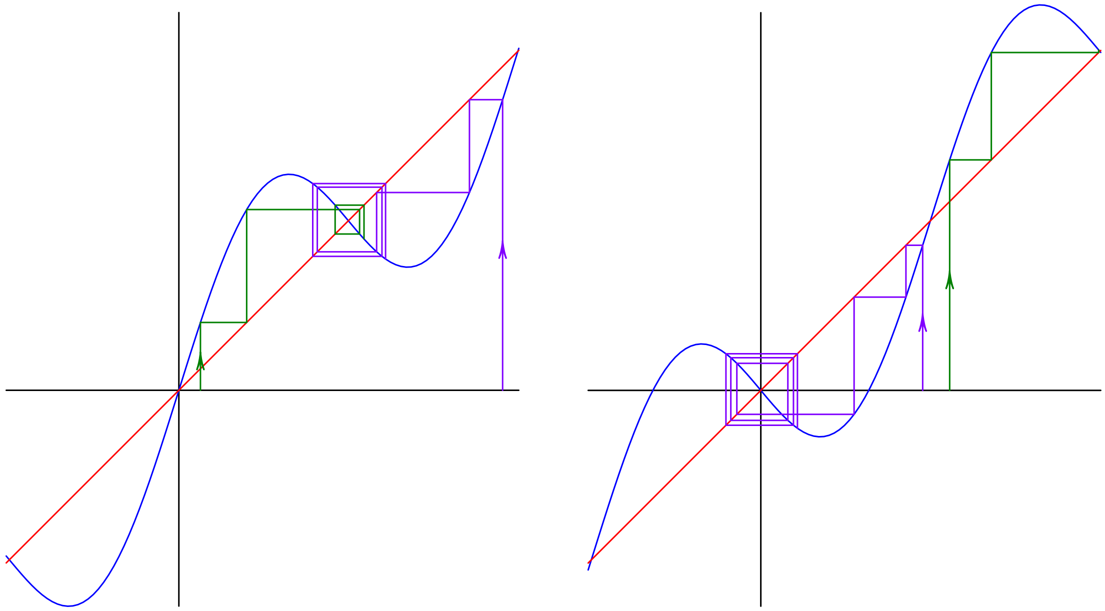
possibilities), and this type of "convergence" would never occur in practice. One of the initial conditions shown in Figure 1(b) almost leads to $x_2 = 2\pi$, but if we zoom in on the region around the fixed point $\alpha = 2\pi$, we would reveal that there is no convergence.



(A) $\kappa = 2.2$ (B) $\kappa = -2.2$

FIGURE 1. Cobweb diagrams for unstable fixed points.

Case $0 < \kappa \leq 1$**.** In this case, the fixed points πm with odd m are stable, and the fixed points πm with even m are unstable. By periodicity, it is sufficient to look at only the interval $[0, 2\pi]$. We observe from Figure 2(a) that the sequence $\{x_n\}$ is monotone. To prove this, first, note that as long as $\kappa > 0$, we have

$$\phi(x) > x \quad \text{for} \quad 0 < x < \pi, \qquad \text{and} \qquad \phi(x) < x \quad \text{for} \quad \pi < x < 2\pi. \tag{2}$$

Thus if the sequence $\{x_n\}$ stays in either one of the intervals $(0, \pi)$ or $(\pi, 2\pi)$, then the sequence would be strictly increasing or decreasing. Moreover,

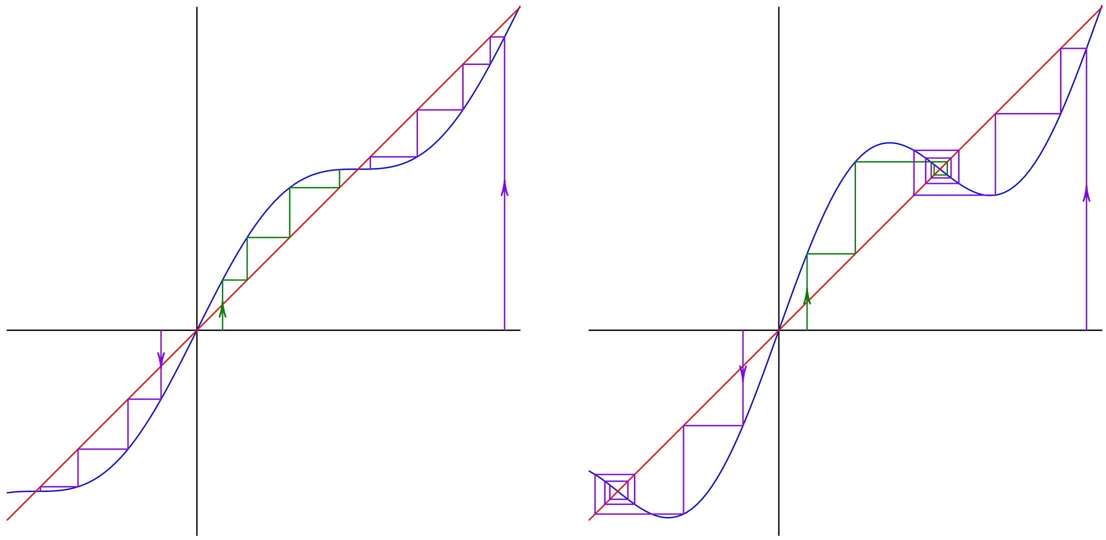
$$\phi'(x) = 1 + \kappa \cos x > 0 \qquad \text{for} \quad x \in (0, \pi) \cup (\pi, 2\pi),$$

that is, ϕ is strictly increasing, provided that $\kappa \leq 1$. Since $\phi(\pi) = \pi$, this shows that

$$x < \phi(x) < \pi \quad \text{for} \quad 0 < x < \pi, \qquad \text{and} \qquad \pi < \phi(x) < x \quad \text{for} \quad \pi < x < 2\pi.$$

In other words, if $x_0 \in (0, \pi)$, then $\{x_n\}$ is a strictly increasing sequence, bounded above by π, and if $x_0 \in (\pi, 2\pi)$, then $\{x_n\}$ is a strictly decreasing sequence, bounded below by π. In either case, the sequence converges, and as we have reasoned earlier, the limit must be a fixed point. However, the only fixed point in the interval $(0, 2\pi)$ is π, and hence $x_n \to \pi$ as $n \to \infty$. From (1), the convergence is linear for $0 < \kappa < 1$, and quadratic for $\kappa = 1$.

Exercise: What happens when $x_0 \in (2k\pi, 2k\pi + 2\pi)$, or $x_0 = 2k\pi$?



(A) $\kappa = 1$ (B) $\kappa = 1.8$

FIGURE 2. Stable fixed points present.

Case $1 < \kappa < 2$**.** As before, let us restrict ourselves to the interval $(0, 2\pi)$. Since $\kappa > 0$, we still have the property (2). Figure 2(b) shows that we no longer have monotonicity of the sequence $\{x_n\}$, and hence we cannot resort to a monotonicity argument as in the case $0 < \kappa \leq 1$.

(a) Note that $\phi''(x) = -\kappa \sin x$. Then since

$$\phi''(\tfrac{\pi}{2}) = -\kappa, \qquad \text{and} \qquad \phi''(\tfrac{3\pi}{2}) = \kappa,$$

there is $\delta_0 > 0$ small enough, such that

$$\phi''(x) \leq -1 \quad \text{for} \quad x \in [\tfrac{\pi}{2}, \tfrac{\pi}{2} + \delta_0], \qquad \text{and} \qquad \phi''(x) \geq 1 \quad \text{for} \quad x \in [\tfrac{3\pi}{2} - \delta_0, \tfrac{3\pi}{2}].$$

This implies that

$$\phi'(x) \leq 1 - \delta \quad \text{for} \quad x \in [\tfrac{\pi}{2} + \delta, \tfrac{3\pi}{2} - \delta],$$

as long as $0 < \delta \leq \delta_0$.

(b) The function $\phi(x)$ is increasing when $x \in [0, \frac{\pi}{2} + \delta]$ provided that $\delta > 0$ is sufficiently small, so if $x \in (0, \frac{\pi}{2} + \delta)$, then

$$\phi(x) \leq \phi(\tfrac{\pi}{2} + \delta) \leq \phi(\tfrac{\pi}{2}) + \delta = \tfrac{\pi}{2} + \kappa + \delta,$$

where we have taken into account that $\phi'(\xi) \leq 1$ for $\xi \in [\frac{\pi}{2}, \frac{\pi}{2} + \delta]$. Now, by choosing $\delta > 0$ small, we can guarantee that $\frac{\pi}{2} + \kappa + \delta < \frac{3\pi}{2}$, so that $x \in (0, \frac{\pi}{2} + \delta)$ implies $\phi(x) < \frac{3\pi}{2}$. What this means is that if $x_0 \in (0, \frac{\pi}{2} + \delta)$, then we have either $x_m \in [\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta]$ for some m, or $x_n < \frac{\pi}{2} + \delta$ for all n. In the latter case, since $\{x_n\}$ is monotonically increasing and bounded, it must converge to some point in $(0, \frac{\pi}{2} + \delta]$. However, the map ϕ has no fixed point in the interval $(0, \frac{\pi}{2} + \delta]$, leading to contradiction. Therefore, if $x_0 \in (0, \frac{\pi}{2} + \delta)$ then $x_m \in [\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta]$ for some m. The case $x_0 \in (\frac{3\pi}{2} - \delta, 2\pi)$ can be treated similarly, and we conclude that for any $x_0 \in (0, 2\pi)$, there is some m such that $x_m \in [\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta]$.

(c) Finally, we want to show that if $x_m \in [\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta]$ for some m, then $x_n \to \pi$ as $n \to \infty$. The minimum of $\phi'(x)$ is attained at $x = \pi$, which is $\phi'(\pi) = 1 - \kappa$. Hence we infer

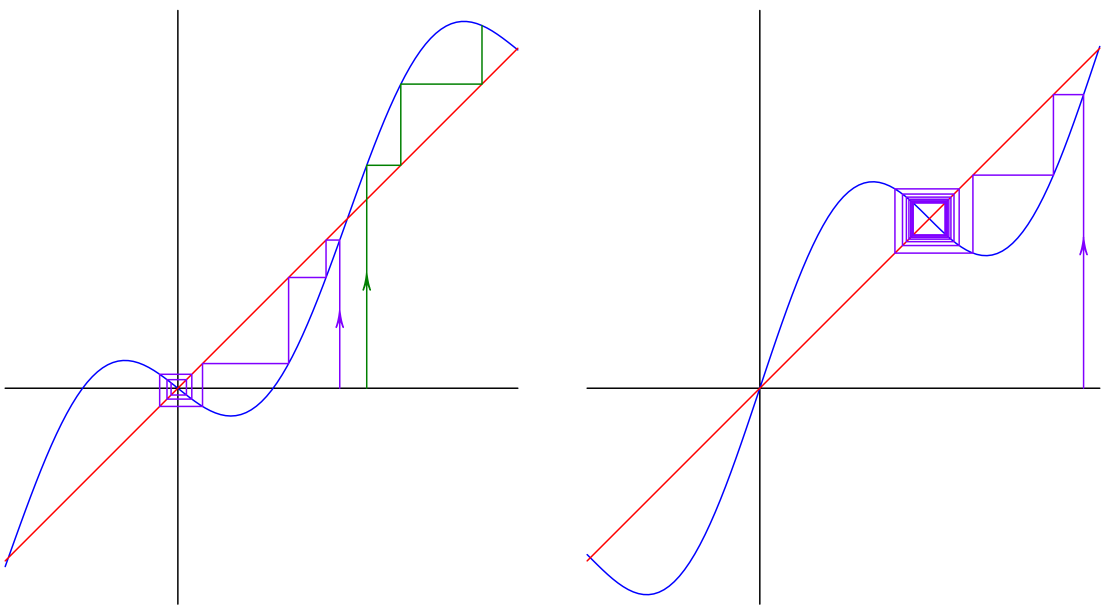
$$|\phi'(x)| \leq \rho := \max\{1 - \delta, \kappa - 1\} < 1 \quad \text{for} \quad x \in [\tfrac{\pi}{2} + \delta, \tfrac{3\pi}{2} - \delta],$$

leading to

$$|\phi(x) - \pi| = \left| \int_\pi^x \phi'(t)\,\mathrm{d}t \right| \leq \rho |x - \pi| \quad \text{for} \quad x \in [\tfrac{\pi}{2} + \delta, \tfrac{3\pi}{2} - \delta].$$

Since $\rho < 1$, we conclude that if $x_m \in [\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta]$ for some m, then $x_n \to \pi$ as $n \to \infty$. As discussed before, the convergence is linear in this case, cf. (1).

Exercise: Treat the case $-2 < \kappa < 0$, cf. Figure 3(a).



(A) $\kappa = -1.8$ (B) $\kappa = 2$

FIGURE 3. Negative κ and the borderline case $\kappa = 2$.

Case $\kappa = 2$. This case is a bit more delicate because $\phi'(\pi) = -1$. If ϕ was simply a line with slope -1, then there would be no convergence. However, in the current situation we have $\phi'(x) = 1 + 2\cos x > -1$ for $x \in (\pi - \varepsilon, \pi) \cup (\pi, \pi + \varepsilon)$ with $\varepsilon > 0$ small. Thus we conjecture that $x_n \to \pi$ as $n \to \infty$, for $x_0 \in (0, 2\pi)$, cf. Figure 3(b).

To prove the conjecture, we can reuse the arguments (a) and (b) from the preceding case ($0 < \kappa < 2$) without any modifications. Namely, we can choose $\delta > 0$ small, as in the preceding case, such that $x_0 \in (0, 2\pi)$ implies $x_m \in [\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta]$ for some m, and that

$$\phi'(x) \leq 1 - \delta \quad \text{for} \quad x \in [\tfrac{\pi}{2} + \delta, \tfrac{3\pi}{2} - \delta].$$

The argument (c) breaks down because $\min \phi' = \phi'(\pi) = -1$. Consider the set

$$K_\varepsilon = [\tfrac{\pi}{2} + \delta, \pi - \varepsilon] \cup [\pi + \varepsilon, \tfrac{3\pi}{2} - \delta].$$

In this set, we have $\phi' > -1$, and so

$$|\phi'(x)| \leq \rho_\varepsilon \quad \text{for} \quad x \in K_\varepsilon,$$

with some $\rho_\varepsilon < 1$, possibly depending on $\varepsilon > 0$. Let $x \in [\pi + \varepsilon, \frac{3\pi}{2} - \delta]$. Then we have

$$|\phi(x) - \pi| \leq \Big| \int_\pi^{\pi+\varepsilon} \phi'(t)\,\mathrm{d}t \Big| + \Big| \int_{\pi+\varepsilon}^{x} \phi'(t)\,\mathrm{d}t \Big| \leq \varepsilon + \rho_\varepsilon |x - \pi - \varepsilon| \leq \rho_\varepsilon |x - \pi| + (1 - \rho_\varepsilon)\varepsilon.$$

Similarly, for $x \in [\frac{\pi}{2} + \delta, \pi - \varepsilon]$, we have

$$|\phi(x) - \pi| \leq \Big| \int_x^{\pi-\varepsilon} \phi'(t)\,\mathrm{d}t \Big| + \Big| \int_{\pi-\varepsilon}^{\pi} \phi'(t)\,\mathrm{d}t \Big| \leq \rho_\varepsilon |\pi - \varepsilon - x| + \varepsilon \leq \rho_\varepsilon |x - \pi| + (1 - \rho_\varepsilon)\varepsilon.$$

Now suppose that $x_m \in I_\varepsilon$. Then we have

$$\begin{aligned}
|x_{m+1} - \pi| &\leq \rho_\varepsilon |x_m - \pi| + (1 - \rho_\varepsilon)\varepsilon, \\
|x_{m+2} - \pi| &\leq \rho_\varepsilon^2 |x_m - \pi| + \rho_\varepsilon(1 - \rho_\varepsilon)\varepsilon + (1 - \rho_\varepsilon)\varepsilon, \\
|x_{m+3} - \pi| &\leq \rho_\varepsilon^3 |x_m - \pi| + \rho_\varepsilon^2(1 - \rho_\varepsilon)\varepsilon + \rho_\varepsilon(1 - \rho_\varepsilon)\varepsilon + (1 - \rho_\varepsilon)\varepsilon, \\
&\dots \\
|x_{m+k} - \pi| &\leq \rho_\varepsilon^k |x_m - \pi| + (1 + \rho_\varepsilon + \ldots + \rho_\varepsilon^{k-1})(1 - \rho_\varepsilon)\varepsilon \leq \rho_\varepsilon^k |x_m - \pi| + \varepsilon,
\end{aligned}$$

and hence $|x_n - \pi| \leq 2\varepsilon$ for all large n. Since this is true for any $\varepsilon > 0$, we conclude that $x_n \to \pi$ as $n \to \infty$.

Exercise: Determine the exact order of convergence for the case $\kappa = 2$.

Exercise: Treat the case $\kappa = -2$.

Problem 4

Consider the polynomial

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n,$$

as a function $p : [0, 1] \to \mathbb{R}$, where $a_0, \ldots, a_n \in \mathbb{R}$. Let $\alpha = p(y)$, with $y \in [0, 1]$ given, and let $\tilde{\alpha} \in \mathbb{R}$ be the result of a computation of $p(y)$ in floating point arithmetic, with the "machine epsilon" $\varepsilon > 0$. Show that there exists a polynomial $\tilde{p}$ of degree at most n, such that $\tilde{p}(y) = \tilde{\alpha}$ in exact arithmetic and that

$$\|p - \tilde{p}\|_\infty = \max_{x \in [0,1]} |p(x) - \tilde{p}(x)| \leq C\varepsilon, \tag{3}$$

for all small $\varepsilon > 0$, where C may depend on n and the coefficients of the polynomial p. Argue that evaluation of polynomials is backward stable, and estimate the error $|\tilde{\alpha} - \alpha|$.

Solution

We consider the following simple algorithm

$$\tilde{\alpha} = a_0 \oplus (a_1 \otimes y) \oplus \ldots \oplus (a_n \otimes y \otimes \cdots \otimes y). \tag{4}$$

Let

$$\tilde{b}_{jk} = a_j \otimes \underbrace{y \otimes \cdots \otimes y}_{k \text{ times}}.$$

Then we have $\tilde{b}_{j,0} = a_j$, and

$$\tilde{b}_{jk} = \tilde{b}_{j,k-1} y(1+\delta_{jk}) = a_j y^k (1+\delta_{j,1}) \cdots (1+\delta_{jk}),$$

where $|\delta_{ji}| \leq \varepsilon$. With these notations, we have

$$\tilde{\alpha} = \tilde{b}_{0,0} \oplus \tilde{b}_{1,1} \oplus \ldots \oplus \tilde{b}_{n,n},$$

and hence

$$\begin{aligned}
\tilde{\alpha} &= (\ldots((\tilde{b}_{0,0} + \tilde{b}_{1,1})(1+\varepsilon_1) + \tilde{b}_{2,2})(1+\varepsilon_2) + \ldots + \tilde{b}_{n,n})(1+\varepsilon_n) \\
&= (\tilde{b}_{0,0} + \tilde{b}_{1,1})(1+\varepsilon_1)\cdots(1+\varepsilon_n) + \tilde{b}_{2,2}(1+\varepsilon_2)\cdots(1+\varepsilon_n) + \ldots + \tilde{b}_{n,n}(1+\varepsilon_n),
\end{aligned}$$

where $|\varepsilon_k| \leq \varepsilon$. Now it is clear that the polynomial

$$\tilde{p}(x) = \tilde{a}_0 + \tilde{a}_1 x + \ldots + \tilde{a}_n x^n,$$

with the coefficients defined by

$$\begin{aligned}
\tilde{a}_0 &= a_0(1+\varepsilon_1)\cdots(1+\varepsilon_n), \\
\tilde{a}_1 &= a_1(1+\delta_{1,1})(1+\varepsilon_1)\cdots(1+\varepsilon_n), \\
\tilde{a}_2 &= a_2(1+\delta_{2,1})(1+\delta_{2,2})(1+\varepsilon_2)\cdots(1+\varepsilon_n), \\
&\ldots \\
\tilde{a}_k &= a_k(1+\delta_{k,1})\cdots(1+\delta_{k,k})(1+\varepsilon_k)\cdots(1+\varepsilon_n), \\
&\ldots \\
\tilde{a}_n &= a_n(1+\delta_{n,1})\cdots(1+\delta_{n,n})(1+\varepsilon_n),
\end{aligned}$$

yields

$$\tilde{\alpha} = \tilde{p}(y) \equiv \tilde{a}_0 + \tilde{a}_1 y + \ldots + \tilde{a}_n y^n.$$

Next, we need to estimate the norm $\|p - \tilde{p}\|_\infty$. We start with

$$\begin{aligned}
|\tilde{a}_k - a_k| &\leq |a_k| \cdot |(1+\delta_{k,1})\cdots(1+\delta_{k,k})(1+\varepsilon_k)\cdots(1+\varepsilon_n) - 1| \\
&\leq |a_k| \cdot |(1+\varepsilon)^{n+1} - 1|.
\end{aligned}$$

Then taking into account

$$(1+\varepsilon)^k = \sum_{i=0}^{k} \frac{k(k-1)\cdots(k-i+1)}{1\cdot 2\cdots i}\varepsilon^i \leq \sum_{i=0}^{k}(k\varepsilon)^i \leq \sum_{i=0}^{\infty}(k\varepsilon)^i = \frac{1}{1-k\varepsilon},$$

which is valid for $k\varepsilon < 1$, we infer

$$|\tilde{a}_k - a_k| \leq \Big(\frac{1}{1-(n+1)\varepsilon} - 1\Big)|a_k| = \frac{(n+1)\varepsilon}{1-(n+1)\varepsilon}|a_k|,$$

for $(n+1)\varepsilon < 1$. Now, for $0 \leq x \leq 1$, we have

$$\begin{aligned}
|\tilde{p}(x) - p(x)| &\leq \sum_{k=0}^{n} |\tilde{a}_k - a_k||x|^k \leq \sum_{k=0}^{n} |\tilde{a}_k - a_k| \\
&\leq \frac{(n+1)\varepsilon}{1-(n+1)\varepsilon}\sum_{k=0}^{n}|a_k| \\
&\leq 2(n+1)\varepsilon \sum_{k=0}^{n}|a_k|,
\end{aligned}$$

for $(n+1)\varepsilon \leq \frac{1}{2}$. This is the desired stability estimate (3) with

$$C = 2(n+1)\sum_{k=0}^{n}|a_k|.$$

In the context where we interpret the evaluation $p(y)$ as a map $p \mapsto p(y) : \mathbb{P}_n \to \mathbb{R}$, *backward stability* means that the computed value $\tilde{\alpha}$ of $p(y)$ can be thought of as the exact evaluation $\tilde{p}(y)$ of a polynomial $\tilde{p}$, where $\tilde{p}$ is within the distance $C\varepsilon$ of p. If we use the uniform norm $\|\cdot\|_\infty$ to measure the distance between polynomials, what we have achieved is exactly backward stability of the polynomial evaluation algorithm (4).

Finally, for the error $|\tilde{\alpha} - \alpha|$, we have the following estimate

$$|\tilde{\alpha} - \alpha| = |\tilde{p}(y) - p(y)| \leq \|\tilde{p} - p\|_\infty \leq 2(n+1)\varepsilon \sum_{k=0}^{n}|a_k|,$$

where $\varepsilon > 0$ and n are assumed to satisfy $(n+1)\varepsilon \leq \frac{1}{2}$.

Problem 5(a)

Find the minimax polynomial approximation of degree 2 for the function $f(x) = \sin x$ on the interval $[-1, 1]$.

Solution

Since $\sin x$ is an odd function on $[-1, 1]$, the minimax polynomial $p(x)$ must also be odd. Moreover, assuming the form $p(x) = ax^2 + bx + c$, we infer that $a = c = 0$. In view of the Chebyshev oscillation theorem, we need to choose the coefficient b such that $f(x) - p(x)$ takes the values $\pm\|f - p\|_\infty$ at least 4 times, with alternating signs. From Figure 4 it is clear that b must be in the range $\sin 1 < b < 1$. Then the local maximums and minimums of $f(x) - p(x)$ are achieved at ± 1 and $\pm\xi$, where $\xi \in (0, 1)$ satisfies $f'(\xi) - p'(\xi) = 0$, i.e.,

$$\cos\xi = b.$$

The condition for $p(x) = bx$ to be the minimax polynomial is now

$$f(-1) - p(-1) = p(-\xi) - f(-\xi) = f(\xi) - p(\xi) = p(1) - f(1).$$

By symmetry, we only need to consider the points $x = \xi$ and $x = 1$, that is,

$$\sin\xi - b\xi = b - \sin 1.$$

In terms of ξ, this becomes

$$(\xi + 1)\cos\xi = \sin\xi + \sin 1,$$

which has a solution $\xi \approx 0.49$. Finally, the minimax polynomial can be written as

$$p(x) = bx = x\cos\xi.$$

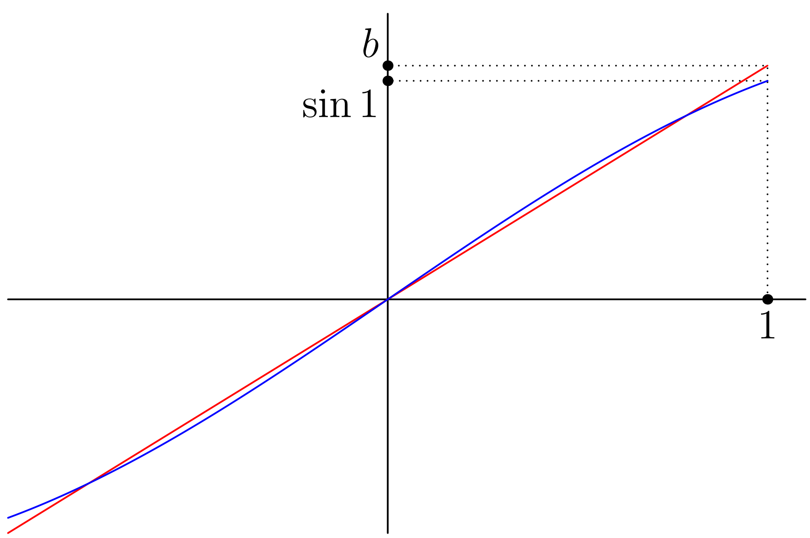


Figure 4. The graphs of $y = \sin x$ (blue) and $y = bx$ (red).

Problem 6(b)

Compute weights and nodes of the quadrature formula

$$\int_0^1 \frac{f(x)\,\mathrm{d}x}{\sqrt{x}} \approx \omega_0 f(x_0) + \omega_1 f(x_1),$$

so that the order of the quadrature is maximum.

Solution

We know from the Gauss-Jacobi theory that the maximum degree of exactness of a quadrature with 2 nodes is 3. Thus we impose

$$\begin{aligned} \omega_0 + \omega_1 &= \int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2, \\ \omega_0 x_0 + \omega_1 x_1 &= \int_0^1 \frac{x\,\mathrm{d}x}{\sqrt{x}} = \frac{2}{3}, \\ \omega_0 x_0^2 + \omega_1 x_1^2 &= \int_0^1 \frac{x^2\mathrm{d}x}{\sqrt{x}} = \frac{2}{5}, \\ \omega_0 x_0^3 + \omega_1 x_1^3 &= \int_0^1 \frac{x^3\mathrm{d}x}{\sqrt{x}} = \frac{2}{7}. \end{aligned} \tag{5}$$

In order to find the nodes x_0 and x_1, let

$$\pi(x) = (x - x_0)(x - x_1) = x^2 + px + q,$$

and derive equations for p and q from (5). First, we multiply the equations in (5) by q, p, 1, and 0, respectively, and sum them, to get

$$2q + \frac{2}{3}p + \frac{2}{5} = 0.$$

Next, by performing the same operation with the coefficients 0, q, p, and 1, we infer

$$\frac{2}{3}q + \frac{2}{5}p + \frac{2}{7} = 0.$$

The solution of this system is

$$p = -\frac{6}{7}, \qquad q = \frac{3}{35},$$

and the roots of $\pi(x) = x^2 + px + q$ are

$$x_0 = \frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}, \qquad x_1 = \frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}.$$

With these nodes, the first two equations in (5) yield

$$\omega_0 = 1 + \frac{1}{3}\sqrt{\frac{5}{6}}, \qquad \omega_1 = 1 - \frac{1}{3}\sqrt{\frac{5}{6}}.$$

Problem 8(d)

Find the least squares approximation polynomials of degrees 0, 1 and 2 for the function $f(x) = |x|$ on the interval $(-1, 1)$ with respect to the weight function $w(x) \equiv 1$.

Solution

Let $p_n \in \mathbb{P}_n$ be the least squares approximation of f from $\mathbb{P}_n$. Then a characteristic property of p_n is that the error $f - p_n$ must be orthogonal to $\mathbb{P}_n$, i.e.,

$$\int_{-1}^{1} (f(x) - p_n(x))x^k \mathrm{d}x = 0 \qquad k = 0, \ldots, n.$$

If we express p_n in terms of the monomial basis $\{1, x, x^2, \ldots\}$, this leads to a Vandermonde type system. A more convenient option is to write p_n in terms of a basis that is orthogonal with respect to the given inner product, in which case one needs to solve a diagonal system. However, for the current problem, we shall use an *ad hoc* approach.

Let us start with the $n = 0$ case. Here, we need to find a constant c such that $f - c$ is orthogonal to all constants (or equivalently, to the constant function 1).

$$\int_{-1}^{1} (f(x) - c)\,\mathrm{d}x = 0 \qquad \Longrightarrow \qquad c\int_{-1}^{1} \mathrm{d}x = \int_{-1}^{1} f(x)\,\mathrm{d}x.$$

This simply means that c is equal to the average of f:

$$p_0(x) \equiv c = \frac{1}{2}\int_{-1}^{1} |x|\,\mathrm{d}x = \frac{1}{2}.$$

Now we consider $n = 1$. Let $p_1(x) = ax + b$. First of all, we require

$$\int_{-1}^{1} (|x| - ax - b)x\,\mathrm{d}x = 0.$$

Since $|x| - b$ is an even function, the integral of $(|x| - a)x$ over $(-1, 1)$ is 0. This means that the integral of ax^2 must be 0, implying that $a = 0$. Thus p_1 is a constant, and since $f - p_1$ must be orthogonal to constants, we conclude that $p_1 = p_0$.

Finally, let $n = 2$ and $p_2(x) = ax^2 + bx + c$. As in the preceding case, a parity argument gives the constraint $b = 0$. The remaining conditions are

$$\int_{-1}^{1} (|x| - ax^2 - c)\,\mathrm{d}x = 0, \qquad \int_{-1}^{1} (|x| - ax^2 - c)x^2 \mathrm{d}x = 0.$$

Upon integration, we get

$$2c + \frac{2}{3}a = 1, \qquad \frac{2}{3}c + \frac{2}{5}a = 1,$$

yielding $a = \frac{15}{4}$ and $c = -\frac{3}{4}$. The conclusion is that

$$p_2(x) = \frac{15x^2 - 3}{4}.$$

See Figure 5 for an illustration.

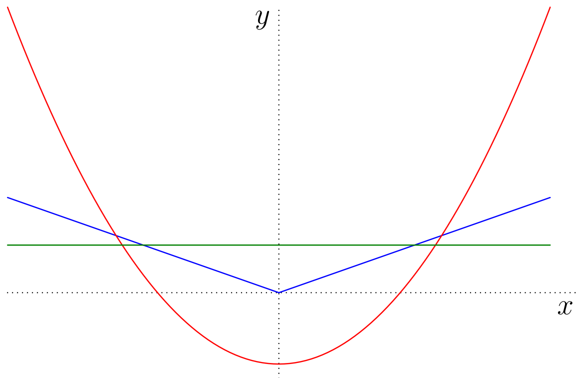


FIGURE 5. The graphs of $y = |x|$ (blue), $y = p_0(x)$ (green), and $y = p_2(x)$ (red).