MATH 387 PRACTICE PROBLEMS

WINTER 2016

1. Analyze the convergence of the fixed point iteration
$$x_{n+1} = x_n + \kappa \sin x_n,$$
for computing the solutions of $\sin(x) = 0$, where $\kappa \neq 0$ is a constant. That is, how do the existence as well as the value of the limit $\lim x_n$ depend on the initial guess x_0, and what is the order of convergence? Of course, the answers will most likely depend on the value of κ. Sketch a cobweb plot of the iterations.
2. Show that the equation $e^x = x + 2$ has two real solutions, $\alpha < 0$ and $\beta > 0$. Letting $x_0, x_1, \ldots$ denote the iterates of the Newton-Raphson method applied to this equation, show that if $x_0 < 0$ then $x_n \to \alpha$ as $n \to \infty$, and if $x_0 > 0$ then $x_n \to \beta$ as $n \to \infty$.
3. Show that the *Steffensen method*
$$x_{n+1} = x_n - \frac{[f(x)]^2}{f(x_n + f(x_n)) - f(x_n)},$$
for solving the equation $f(x) = 0$ converges quadratically, if the initial guess x_0 is sufficiently close to a solution.
4. Consider the polynomial
$$p(x) = a_0 + a_1 x + \ldots + a_n x^n,$$
as a function $p : [0, 1] \to \mathbb{R}$, where $a_0, \ldots, a_n \in \mathbb{R}$. Let $\alpha = p(y)$, with $y \in [0, 1]$ given, and let $\tilde{\alpha} \in \mathbb{R}$ be the result of a computation of $p(y)$ in floating point arithmetic, with the "machine epsilon" $\varepsilon > 0$. Show that there exists a polynomial $\tilde{p}$ of degree at most n, such that $\tilde{p}(y) = \tilde{\alpha}$ in exact arithmetic and that
$$\|p - \tilde{p}\|_\infty = \max_{x \in [0,1]} |p(x) - \tilde{p}(x)| \leq C\varepsilon,$$
for all small $\varepsilon > 0$, where C may depend on n and the coefficients of the polynomial p. Argue that evaluation of polynomials is backward stable, and estimate the error $|\tilde{\alpha} - \alpha|$.
5. In each of the following cases, find the minimax polynomial approximation of degree n for the function $f(x)$ on the interval $[a, b]$. You need to prove that the polynomial you found is indeed the minimax polynomial.
(a) $f(x) = \sin x$, $[a, b] = [-1, 1]$, $n = 2$.
(b) $f(x) = \cos x^2$, $[a, b] = [-1, 1]$, $n = 3$.
(c) $f(x) = |x|$, $[a, b] = [-1, 2]$, $n = 1$.

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6. Compute weights and nodes of the quadrature formula

$$\int_0^1 f(x)w(x)\,\mathrm{d}x \approx \omega_0 f(x_0) + \omega_1 f(x_1),$$

so that the order of the quadrature is maximum, where the weight function is
(a) $w(x) = \log \frac{1}{x}$.
(b) $w(x) = \frac{1}{\sqrt{x}}$.
7. Construct orthogonal polynomials of degrees 0, 1, and 2 on the interval $(0, 1)$ with respect to the weight function
(a) $w(x) = \log \frac{1}{x}$.
(b) $w(x) = \frac{1}{\sqrt{x}}$.
8. In each of the following cases, find the least squares approximation polynomials of degrees 0, 1 and 2 for the function $f(x)$ on the interval (a, b) with respect to the weight function $w(x) \equiv 1$.
(a) $f(x) = \sin x$, $(a, b) = (-\pi, \pi)$.
(b) $f(x) = \sin x$, $(a, b) = (-\frac{\pi}{2}, \frac{\pi}{2})$.
(c) $f(x) = \sin x$, $(a, b) = (0, \pi)$.
(d) $f(x) = |x|$, $(a, b) = (-1, 1)$.