MATH 387 TAKE HOME FINAL EXAM

DUE WEDNESDAY APRIL 27, 23:00 EDT

Notes and instructions

- Please submit by email a single PDF file, typed in LaTeX.
- The subject line of the email should be:

  `[Math 387 Final Exam] %name %n`

  where `%name` is your name, and `%n` is the version number. The latest version is the version to be graded.
- Clearly identify your name in the document.
- The grading will be based on quality of presentation, completeness, correctness, and creativity/resourcefulness.
- Pay special attention to quality and style of presentation. Try to write a story, rather than a list of disconnected items. Low quality presentations will not be accepted and will receive a grade of 0.
- The following languages are allowed in the coding part: `C`, `C++`, `Java`, `Matlab`. You should contact the instructor if you want to use a language that is not listed here.

Background material

In this exercise, we will perform an experimental study of the Lebesgue constants for various interpolation, least squares approximation, and quadrature processes.

Recall that the *Lebesgue constants for Lagrange interpolation* are defined as

$$\|L_n\| = \sup_{f\in C([a,b])} \frac{\|L_n f\|_\infty}{\|f\|_\infty},$$

where $L_n f \in \mathbb{P}_n$ is the Lagrange interpolation polynomial for f, associated to the given set of (distinct) nodes $x_0, \ldots, x_n$. The Lebesgue constants are of importance because

(i) We have

$$\|f - L_n f\|_\infty \leq (1 + \|L_n\|) \inf_{q\in\mathbb{P}_n} \|f - q\|_\infty,$$

which gives information on how the interpolation compares to the minimax approximation, and how we can use the theory of minimax approximation (Jackson's theorem, Taylor expansions, etc.) to derive error estimates for interpolation.

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(ii) $\|f - L_n f\|_\infty \to 0$ as $n \to \infty$ implies that $\sup_n \|L_n\| < \infty$. So if the Lebesgue constants are not uniformly bounded, there is a function $f \in C([a,b])$ such that $L_n f \not\to f$ as $n \to \infty$.
(iii) A large Lebesgue constant would indicate numerical instability (i.e., amplification of round-off errors).

Consider the inner product and the corresponding norm

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)\mathrm{d}x, \qquad \text{and} \qquad \|f\| = \sqrt{\langle f, f\rangle},$$

respectively, for functions defined on (a,b), where $w \in C((a,b))$ is a (positive) weight function. For $f \in C((a,b))$ with $\|f\| < \infty$, let $S_n f \in \mathbb{P}_n$ be the least-squares approximation of f with respect to the norm $\|\cdot\|$, i.e., $S_n f$ is the orthogonal projection of f onto $\mathbb{P}_n$. Then the *Lebesgue constant for the least-squares approximation* is

$$\|S_n\| = \sup_{f \in C([a,b])} \frac{\|S_n f\|_\infty}{\|f\|_\infty}.$$

In terms of the Lebesgue constants, we have the error estimate

$$\|f - S_n f\|_\infty \leq (1 + \|S_n\|) \inf_{q \in \mathbb{P}_n} \|f - q\|_\infty,$$

for the least-squares approximations, and other results analogous to those for interpolation. Given a sequence $\phi_0, \phi_1, \ldots$ of polynomials, such that $\{\phi_0, \ldots, \phi_n\}$ is an orthonormal basis of of $\mathbb{P}_n$ with respect to the inner product $\langle\cdot,\cdot\rangle$ for each n, we have

$$S_n f = \sum_{k=0}^{n} \langle f, \phi_k\rangle \phi_k.$$

In this context, $S_n f$ is also called a *truncation* of the series of f in terms of the polynomials $\{\phi_k\}$. For example, we can talk about *Legendre truncation* or *Chebyshev truncation*, depending on the polynomials under consideration.

Finally, consider the quadrature formula

$$Q_n(f) = \sum_{k=0}^{n} \omega_k f(x_k),$$

that is meant to approximate the integral

$$I(f) = \int_a^b f(x)w(x)\mathrm{d}x,$$

where $w > 0$ is a given weight function. Assuming that $Q_n(f) = I(f)$ for $f \in \mathbb{P}_m$, we can derive the error estimate

$$|Q_n(f) - I(f)| \leq (\|I\| + \|Q_n\|) \inf_{q \in \mathbb{P}_m} \|f - q\|_\infty,$$

where

$$\|I\| = \sup_{f\in C([a,b])} \frac{|I(f)|}{\|f\|_\infty} = \int_a^b w(x)\mathrm{d}x,$$

and

$$\|Q_n\| = \sup_{f\in C([a,b])} \frac{|Q_n(f)|}{\|f\|_\infty}.$$

Therefore $\|Q_n\|$ plays the role of Lebesgue constants for quadrature formulas. It is easy to see that $\|Q_n\| = \|I\|$ if the weights ω are positive and the degree of exactness satisfies $m \geq 0$. So the constants $\|Q_n\|$ may potentially grow with n only if negative weights occur in the quadrature formula.

The exercise

(a) Show that

$$\|L_n\| = \max_{x\in[a,b]} \sum_{k=0}^{n} |\phi_{n,k}(x)|, \tag{1}$$

where $\phi_{n,k}$ is the k-th Lagrange coefficient associated to the nodes $x_0, \ldots, x_n$.

(b) For Chebyshev truncation, we have derived in class that

$$\|S_n\| = \int_0^\pi \Big|\frac{\sin((n+\frac{1}{2})\theta)}{2\sin(\frac{1}{2}\theta)}\Big|\mathrm{d}\theta. \tag{2}$$

Derive an expression of the form

$$\|S_n\| = \int_{-1}^{1} \Big|\sum_{k=0}^{n} a_k P_k(x)\Big|\mathrm{d}x, \tag{3}$$

for the Lebesgue constants of the Legendre truncation, where P_k are the Legendre polynomials.

(c) For interpolatory quadrature with the nodes $x_0, \ldots, x_n$, for approximating the integral over (a,b) with weight $w(x) = 1$, show that

$$\|Q_n\| = \sum_{k=0}^{n} |\omega_k| = \sum_{k=0}^{n} \Big|\int_a^b \phi_{n,k}(x)\mathrm{d}x\Big|, \tag{4}$$

where $\phi_{n,k}$ is the k-th Lagrange coefficient associated to the nodes $x_0, \ldots, x_n$.

(d) For Chebyshev interpolation (cf. (1)), the function

$$\lambda_n(x) = \sum_{k=0}^{n} |\phi_{n,k}(x)|,$$

takes its maximum at $x = \pm 1$, so in order to compute the Lebesgue constant $\|L_n\|$, we only need to be able to evaluate the Lagrange coefficients $\phi_{n,k}$. Plotting the graph (of a function) of $\|L_n\|$ against n, experimentally determine the constant C in the assumed dependence $\|L_n\| = C\log n$.

(e) For interpolation with equidistant nodes, the function λ_n as above achieves its maximum in the "outermost" subintervals, that is, in the intervals (x_0, x_1) and (x_{n-1}, x_n). So in order to compute the Lebesgue constant $\|L_n\|$, we need to locate one of the global maximums. Implement Newton's method and apply it to the derivative λ_n' to complete this task. One possible initial guess is to start at the midpoint of the interval (x_0, x_1). Make sure that Newton's method converges to the desired zero of λ_n'. Plot the graph (of a function) of $\|L_n\|$ against n, and experimentally confirm the law $\|L_n\| \sim 2^n$.

(f) To compute the Lebesgue constants for Chebyshev truncation, we need to evaluate the integrals (2). Design and implement composite trapezoidal or Simpson's rule for this task. When you design the quadrature rule, make sure that you take into account the behaviour of the integrand especially for large n. For example, one could choose the subintervals of the composite quadrature rule to be aligned with the zeroes of the integrand. Plotting the graph (of a function) of $\|S_n\|$ against n, experimentally determine the constant C in the assumed dependence $\|S_n\| = C \log n$.

(g) Design and implement a quadrature rule for the integrals (3), to compute the Lebesgue constants for Legendre truncation. Plotting the graph (of a function) of $\|S_n\|$ against n, experimentally determine the constant C in the assumed dependence $\|S_n\| = C\sqrt{n}$.

(h) We want to show some evidence that for Newton-Cotes formulas, $\|Q_n\|$ grows with n. This would indicate that $Q_n(f) \not\to I(f)$ as $n \to \infty$. Design and implement a procedure to compute the weights ω_k, and therefore the "Lebesgue constant" $\|Q_n\|$ for the Newton-Cotes formula with $n+1$ nodes, for the interval, say $[0, 1]$. This can be done by a quadrature rule for the integrals in (4), or by solving a Vandermonde type system. If you choose the latter, make sure that you do not run into a numerical instability problem due to the conditioning of the matrices. Plotting the graph (of a function) of $\|Q_n\|$ against n, make a guess on the functional dependence of $\|Q_n\|$ on n.