MATH 387 ASSIGNMENT 4

DUE THURSDAY APRIL 14

1. Let $\rho \in C(\mathbb{R})$ be a *nonnegative* function satisfying
$$\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = 1, \qquad \text{and} \qquad \rho(x) = 0 \quad \text{for } |x| > 1.$$
For example, one can take
$$\rho(x) = \max\{0, 1 - |x|\}.$$
Then for $\varepsilon > 0$, define
$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(x/\varepsilon).$$
Note that ρ_ε satisfies
$$\int_{\mathbb{R}} \rho_\varepsilon(x)\,\mathrm{d}x = 1, \qquad \text{and} \qquad \rho_\varepsilon(x) = 0 \quad \text{for } |x| > \varepsilon.$$
Now, suppose that $x_0, x_1 \ldots, x_n$ are distinct points in some interval (a, b), and consider the weight function
$$w_\varepsilon(x) = \rho_\varepsilon(x - x_0) + \rho_\varepsilon(x - x_1) + \ldots + \rho_\varepsilon(x - x_n),$$
for small $\varepsilon > 0$. Let $f \in C([a, b])$, and let $p_\varepsilon \in \mathbb{P}_n$ be the least-squares approximation of f with respect to the weight w_ε. Informally speaking, the weight w_ε tries to drive the approximation to be accurate in the regions near the nodes $x_0, x_1 \ldots, x_n$. Show that
$$\|p_\varepsilon - p\|_\infty \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$
where $p \in \mathbb{P}_n$ is the Lagrange interpolation polynomial of f with the nodes $\{x_0, x_1 \ldots, x_n\}$.
2. For functions $f \in C([a, b])$ where $-\infty < a < b < \infty$, and for $1 < p < \infty$, define the *p-norm*
$$\|f\|_p = \Big(\int_a^b |f(x)|^p \mathrm{d}x \Big)^{\frac{1}{p}},$$
and consider the problem of approximating f by polynomials in the p-norm: Find $q \in \mathbb{P}_n$ such that $\|f - q\|_p$ is minimal.
(a) Show that for any $f \in C([a, b])$, there exists $g_n \in \mathbb{P}_n$ such that
$$\|f - g_n\|_p = \inf_{q \in \mathbb{P}_n} \|f - q\|_p.$$
(b) Show that the best approximation $g_n \in \mathbb{P}_n$ as in (a) is unique.
(c) Show that g_n converges to f in the p-norm as $n \to \infty$.

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(d) Design an algorithm to compute g_n.

3. Consider the inner product and the corresponding norm

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)e^{-x^2}\mathrm{d}x, \qquad \text{and} \qquad \|f\| = \sqrt{\langle f, f\rangle},$$

respectively, for functions defined on $\mathbb{R} = (-\infty, \infty)$. Starting with the monomials $1, x, x^2, \ldots$, one can generate orthogonal polynomials with respect to the inner product $\langle\cdot,\cdot\rangle$. Up to a normalization, these are called the *Hermite polynomials*.

(a) Compute the first 6 Hermite polynomials, with the normalization that the leading coefficient of the n-th degree Hermite polynomial is 2^n.

(b) Let $f \in C(\mathbb{R})$, and suppose that for some m,

$$\sup_{x\in\mathbb{R}} \frac{|f(x)|}{1+|x|^m} < \infty.$$

In other words, f grows slower than a polynomial at infinity. Show that there exists a unique $g_n \in \mathbb{P}_n$ such that

$$\|f - g_n\| = \inf_{q\in\mathbb{P}_n} \|f - q\|.$$

(c) Show that $\|f - g_n\| \to 0$ as $n \to \infty$, where f and g_n are as in (b).

4. In each of the following cases, compute weights and nodes of the quadrature formula

$$\int_a^b w(x)f(x)\,\mathrm{d}x \approx \omega_0 f(x_0) + \omega_1 f(x_1) + \ldots + \omega_n f(x_n),$$

so that the order (or equivalently, the degree of exactness) of the quadrature is maximum.

(a) $w(x) = 1 + \theta(x)$, $(a, b) = (-1, 1)$, $n = 1$, where θ is the Heaviside step function.

(b) $w(x) = \sin x$, $(a, b) = (0, \frac{\pi}{2})$, $n = 1$.

(c) $w(x) = e^{-x}$, $(a, b) = (0, \infty)$, $n = 3$.

5. In each of the following cases, analyze the convergence of the fixed point iteration

$$x_{n+1} = \phi(x_n),$$

for computing the solutions of $f(x) = 0$. That is, how do the existence as well as the value of the limit $\lim x_n$ depend on the initial guess x_0, and what is the order of convergence? Sketch a cobweb plot of the iteration.

(a) $\phi(x) = \cos x$, $f(x) = x - \cos x$.

(b) $\phi(x) = x^2 - 2$, $f(x) = x^2 - x - 2$.

(c) $\phi(x) = -\sqrt{x+2}$, $f(x) = x^2 - x - 2$.

(d) $\phi(x) = x - 2 + \frac{x}{x-1}$, $f(x) = \frac{2x^2-3x-2}{x-1}$.

6. In each of the following cases, propose two different fixed point methods for approximating the root $x = \alpha$ of $f(x) = 0$, such that one method is linearly convergent, and the other is quadratically convergent. Give detailed proofs of convergence.

(a) $f(x) = e^{-x} - \sin x$, and α is the smallest positive root.

(b) $f(x)$ has a double root at $x = \alpha$.