1. Let \( f(z) = y - 2xy + i(-x + x^2 - y^2) + z^2 \) where \( z = x + iy \) is a complex variable defined in the whole complex plane. For what values of \( z \) does \( f'(z) \) exist?

Solution: Our plan is to identify the real and imaginary parts of \( f \), and then check if the Cauchy-Riemann equations hold for them. We have
\[
f(z) = y - 2xy + i(-x + x^2 - y^2) + x^2 - y^2 + 2ixy = x^2 - 2xy + y - y^2 + i(-x + 2xy + x^2 - y^2),
\]
and so
\[
u(x, y) = x^2 - 2xy + y - y^2, \quad v(x, y) = -x + 2xy + x^2 - y^2.
\]
We compute the partial derivatives of \( u \) and \( v \) as
\[
u_x(x, y) = 2x - 2y,
\]
\[
u_y(x, y) = -2x + 1 - 2y,
\]
\[
u_x(x, y) = -1 + 2y + 2x,
\]
\[
u_y(x, y) = 2x - 2y.
\]
We see that the Cauchy-Riemann equations
\[
u_x = v_y, \quad v_x = -u_y,
\]
hold all \( x \) and \( y \), which means that \( f'(z) \) exists for all values of \( z \), i.e., the function \( f \) is an entire function. For completeness, we can compute the derivative
\[
f'(z) = u_x + iv_x = 2x - 2y + i(2x + 2y - 1) = 2z + 2iz - i.
\]

Alternative solution: Another way to solve this would be to notice that
\[
f(z) = z^2 + iz^2 - iz,
\]
which reveals that \( f \) is entire since \( f \) is a polynomial (in \( z \)).

2. In the preceding question, take \( f(z) = \cos x - i \sinh y \).

Solution: This time the Cauchy-Riemann equations are quicker:
\[
(\cos x)'_x = -\sin x, \quad (\sinh y)'_x = 0,
\]
\[
(\cos x)'_y = 0, \quad (\sinh y)'_y = -\cosh y.
\]
The equation \(-\sin x = -\cosh y\) is never satisfied because \( \sin x \leq 1 \) and \( \cosh y > 1 \) except at \( y = 0 \). So \( f'(z) \) does not exist anywhere.
3. Show that \( f(z) = (\bar{z} + 1)^3 - 3\bar{z} \) is nowhere analytic.

Solution: Expanding the cubic, we get

\[
f(z) = (x + 1 - yi)^3 - 3(x - yi) = (x + 1)^3 - 3(x + 1)y^2 - 3(x + 1)^2yi + y^3i - 3x + 3yi
\]

Let us compute the partial derivatives of \( u \) and \( v \).

\[
\begin{align*}
u_x &= 3(x + 1)^2 - 3y^2 - 3, & v_x &= -6(x + 1)y, \\
u_y &= -6(x + 1)y, & v_y &= 3y^2 + 3 - 3(x + 1)^2.
\end{align*}
\]

We see that they satisfy exactly the opposite of what we want. For instance, we have \( v_x = u_y \), rather than \( v_x = -u_y \). So \( v_x = -u_y \) holds, only when \( v_x = u_y = 0 \). This means that \( 6(x + 1)y = 0 \), i.e., \( x = -1 \) or \( y = 0 \). Therefore \( f(z) \) has a chance of being differentiable only at the lines \( x = -1 \) and \( y = 0 \). But then \( f(z) \) cannot be analytic, as any neighbourhood of each point on those lines will contain a point not on any of the lines, at which \( f \) is not differentiable.

4. Find \( \frac{d^5(e^{2t} \sin(2t))}{dt^5} \).

Solution: Since \( \sin(2t) \) is the imaginary part of \( e^{2it} \), \( e^{2t} \sin(2t) \) is the imaginary part of \( e^{2t(1+i)} \). Hence we can differentiate the latter 5 times and then take the imaginary part of the result to find what is asked. We compute

\[
\frac{d^5(e^{2t(1+i)})}{dt^5} = 2^5(1+i)^5 e^{2t(1+i)}.
\]

In order to take the 5-th power of \( 1 + i \), we write it in polar form as \( 1 + i = \sqrt{2}e^{i\pi/4} \), and compute

\[
(1 + i)^5 = 2^{5/2}e^{5i\pi/4} = -2^{5/2}2^{-1/2}(1 + i),
\]

resulting in

\[
2^5(1 + i)^5 e^{2t(1+i)} = -2^5e^{2t}(1 + i)e^{2t}(\cos 2t + i \sin 2t).
\]

The imaginary part of this is \( -2^7e^{2t}(\cos 2t + \sin 2t) \), i.e.,

\[
\frac{d^5(e^{2t} \sin(2t))}{dt^5} = -2^7e^{2t}(\cos 2t + \sin 2t).
\]

5. What is the value of the integer \( n \) if \( x^n - y^n \) is harmonic?

Solution: Recall that a function \( u(x, y) \) is called harmonic if \( u_{xx} + u_{yy} = 0 \). So \( x^n - y^n \) is harmonic if \( n(n - 1)(x^{n-2} - y^{n-2}) = 0 \), which means that \( n = 0, n = 1, \) or \( n = 2 \).

6. Find the harmonic conjugate of \( e^x \cos y + e^y \cos x + xy \).

Solution: With \( u(x, y) = e^x \cos y + e^y \cos x + xy \), we need to find a function \( v(x, y) \) such that \( f = u + iv \) is analytic, that is, \( u_x = v_y \) and \( v_y = -u_x \). We have

\[
\begin{align*}
u_x &= e^x \cos y - e^y \sin x + y, & v_y &= -e^x \sin y + e^y \cos x + x.
\end{align*}
\]
Integrating $u_x$ with respect to $y$, we get
\[ v(x, y) = e^x \sin y - e^y \sin x + \frac{1}{2} y^2 + A(x), \]
where $A(x)$ is an arbitrary function of $x$. On the other hand, integrating $-u_y$ with respect to $x$, we have
\[ v(x, y) = e^x \sin y - e^y \sin x + \frac{1}{2} x^2 + B(y), \]
where $B(y)$ is an arbitrary function of $y$. Combining the two expressions, we conclude that
\[ v(x, y) = e^x \sin y - e^y \sin x + \frac{1}{2} x^2 + \frac{1}{2} y^2, \]
satisfies $u_x = v_y$ and $v_y = -u_y$.

7. Compute $\arcsin(1 + i)$.

Solution: Recall the formula
\[ \arcsin z = -i \log (zi \pm \sqrt{1 - z^2}). \]

With $z = 1 + i$, we get
\[ \sqrt{1 - z^2} = \sqrt{1 - 2i} = \sqrt{5} \left( \cos \frac{\alpha}{2} - i \sin \frac{\alpha}{2} \right), \]
where $\alpha = \arctan 2$, and so
\[ w_{1,2} = zi \pm \sqrt{1 - z^2} = -1 + i \pm \sqrt{5} \left( \cos \frac{\alpha}{2} - i \sin \frac{\alpha}{2} \right). \]

Now we recall
\[ \log w = \log |w| + i \arg w. \]

Note that $\arg w$ has infinitely many values differing from each other by integer multiples of $2\pi$. Hence each of $w_1$ and $w_2$ results in an infinite collection of values for $\arcsin$. We cannot do much beyond this, except to say that we can compute an approximate value by a calculator.

8. Prove that $\sin(iz) = i \sinh z$ and $\cos(iz) = \cosh z$.

Solution: They follow from the definitions. For instance, the first identity is
\[ \sin(iz) = \frac{e^{i\cdot iz} - e^{-i\cdot iz}}{2i} = \frac{e^{-z} - e^z}{2i} = i \frac{e^z - e^{-z}}{2} = i \sinh z. \]

9. Find all solutions of $\sin z - \cos z = 0$.

Solution: The equation says that
\[ \frac{e^z - e^{-iz}}{2i} - \frac{e^{iz} + e^{-iz}}{2} = \frac{(1 - i)e^{iz} - (1 + i)e^{-iz}}{2i} = 0. \]

Multiplying it by $2ie^{iz}$, we arrive at
\[ (1 - i)e^{2iz} - (1 + i) = 0, \quad \text{or} \quad e^{2iz} = \frac{1 + i}{1 - i} = i. \]

This implies that $2iz = i(\frac{\pi}{2} + 2\pi k)$ for some integer $k$, i.e.,
\[ z = \frac{\pi}{4} + \pi k. \]
10. Compute \( \log e^i \).

*Solution*: We have

\[
z = e^i = e^{\cos 1 + i \sin 1} = e^{\cos 1}(\cos \sin 1 + i \sin \sin 1),
\]

hence \(|z| = e^{\cos 1}\) and \(\arg z = \sin 1 + 2\pi k\), with integers \(k\). We conclude

\[
\log z = \log |z| + i \arg z = \cos 1 + i(\sin 1 + 2\pi k).
\]

11. Find all solutions of \( e^z = e^{iz} \).

*Solution*: The exponential function has the period \(2\pi i\), so

\[
z = iz + 2\pi ik\text{ for some integer }k.
\]

In other words,

\[
z = \frac{2\pi ik}{1 - i} = \left(1 + i\right)\pi k = (1 + i)\pi k.
\]

12. Compute \( \pi^i \).

*Solution*: We compute

\[
\pi^i = e^{i \log \pi} = e^{i(\log \pi + i 2\pi k)} = e^{\pi \log \pi - 2\pi k} = e^{-2\pi k} (\cos \log \pi + i \sin \log \pi).
\]

13. Find all solutions of \( \sin \cos z = 0 \).

*Solution*: Solving the outer equation gives \( \cos z = \pi k \), for some integer \(k\). Then taking the arccosine of it, we have

\[
z = \arccos(\pi k) = -i \log(\pi k \pm \sqrt{\pi^2 k^2 - 1}).
\]

For \(k \neq 0\), we have \(\pi^2 k^2 > 1\), so

\[
z = -i \log(\pi k \pm \sqrt{\pi^2 k^2 - 1}) = -i \Log(\pi k \pm \sqrt{\pi^2 k^2 - 1}) + 2\pi n,
\]

for some integer \(n\). The case \(k = 0\), on the other hand, leads to

\[
z = -i \log(\pm i) = -i \cdot i(\frac{\pi}{2} + 2\pi n) = \frac{\pi}{2} + 2\pi n,
\]

for some integer \(n\).

14. Evaluate \( \int e^z \, dz \), from \(z = 1\) to \(z = 1 + i\) along the line \(x = 1\).

*Solution*: Let us parameterize the line by \(z(t) = 1 + it\), with \(0 \leq t \leq 1\). By definition of the complex line integral, we have

\[
\int e^z \, dz = \int_0^1 e^{z(t)} z'(t) \, dt = \int_0^1 e^{1 + it} \, i \, dt = e^{1 + it} \bigg|_0^1 = e(e^i - 1).
\]

15. Evaluate \( \int \frac{dz}{z} \), from \(-i\) to \(i\) along the arc given by \(z(t) = e^{it}\) with \(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\).

*Solution*: Again by definition, we have

\[
\int \frac{dz}{z} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{z'(t) \, dt}{z(t)} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ie^{it} \, dt}{e^{it}} = it \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi i.
\]