SAMPLE PROBLEMS WITH SOLUTIONS

FALL 2012

1. Let $f(z) = y - 2xy + i(-x + x^2 - y^2) + z^2$ where $z = x + iy$ is a complex variable defined in the whole complex plane. For what values of z does $f'(z)$ exist?

   Solution: Our plan is to identify the real and imaginary parts of f, and then check if the Cauchy-Riemann equations hold for them. We have

$$\begin{aligned} f(z) &= y - 2xy + i(-x + x^2 - y^2) + x^2 - y^2 + 2ixy \\ &= x^2 - 2xy + y - y^2 + i(-x + 2xy + x^2 - y^2), \end{aligned}$$

   and so

$$u(x, y) = x^2 - 2xy + y - y^2, \qquad v(x, y) = -x + 2xy + x^2 - y^2.$$

   We compute the partial derivatives of u and v as

$$\begin{aligned} u_x(x, y) &= 2x - 2y, & v_x(x, y) &= -1 + 2y + 2x, \\ u_y(x, y) &= -2x + 1 - 2y, & v_y(x, y) &= 2x - 2y. \end{aligned}$$

   We see that the Cauchy-Riemann equations

$$u_x = v_y, \qquad v_x = -u_y,$$

   hold all x and y, which means that $f'(z)$ exists for all values of z, i.e., the function f is an entire function. For completeness, we can compute the derivative

$$f'(z) = u_x + iv_x = 2x - 2y + i(2x + 2y - 1) = 2z + 2iz - i.$$

   Alternative solution: Another way to solve this would be to notice that

$$f(z) = z^2 + iz^2 - iz,$$

   which reveals that f is entire since f is a polynomial (in z).

2. In the preceding question, take $f(z) = \cos x - i \sinh y$.

   Solution: This time the Cauchy-Riemann equations are quicker:

$$\begin{aligned} (\cos x)'_x &= -\sin x, & (-\sinh y)'_x &= 0, \\ (\cos x)'_y &= 0, & (-\sinh y)'_y &= -\cosh y. \end{aligned}$$

   The equation $-\sin x = -\cosh y$ is never satisfied because $\sin x \le 1$ and $\cosh y > 1$ except at $y = 0$. So $f'(z)$ does not exist anywhere.

3. Show that $f(z) = (\bar{z}+1)^3 - 3\bar{z}$ is nowhere analytic.

Solution: Expanding the cubic, we get

$$\begin{aligned} f(z) &= (x+1-yi)^3 - 3(x-yi) \\ &= (x+1)^3 - 3(x+1)y^2 - 3(x+1)^2yi + y^3i - 3x + 3yi \\ &= \underbrace{(x+1)^3 - 3(x+1)y^2 - 3x}_{u(x,y)} + i\,\underbrace{(y^3 + 3y - 3(x+1)^2y)}_{v(x,y)}. \end{aligned}$$

Let us compute the partial derivatives of u and v.

$$\begin{aligned} u_x &= 3(x+1)^2 - 3y^2 - 3, & v_x &= -6(x+1)y, \\ u_y &= -6(x+1)y, & v_y &= 3y^2 + 3 - 3(x+1)^2. \end{aligned}$$

We see that they satisfy exactly the opposite of what we want. For instance, we have $v_x = u_y$, rather than $v_x = -u_y$. So $v_x = -u_y$ holds, only when $v_x = u_y = 0$. This means that $6(x+1)y = 0$, i.e, $x = -1$ or $y = 0$. Therefore $f(z)$ has a chance of being differentiable only at the lines $x = -1$ and $y = 0$. But then $f(z)$ cannot be analytic, as any neighbourhood of each point on those lines will contain a point not on any of the lines, at which f is not differentiable.

4. Find $\frac{\mathrm{d}^5(e^{2t}\sin(2t))}{\mathrm{d}t^5}$.

Solution: Since $\sin(2t)$ is the imaginary part of e^{2it}, $e^{2t}\sin(2t)$ is the imaginary part of $e^{2t(1+i)}$. Hence we can differentiate the latter 5 times and then take the imaginary part of the result to find what is asked. We compute

$$\frac{\mathrm{d}^5 e^{2t(1+i)}}{\mathrm{d}t^5} = 2^5(1+i)^5 e^{2t(1+i)}.$$

In order to take the 5-th power of $1+i$, we write it in polar form as $1 + i = \sqrt{2}e^{i\pi/4}$, and compute

$$(1+i)^5 = 2^{5/2}e^{5\pi i/4} = -2^{5/2}2^{-1/2}(1+i),$$

resulting in

$$2^5(1+i)^5 e^{2t(1+i)} = -2^5 2^2(1+i)e^{2t}(\cos 2t + i \sin 2t).$$

The imaginary part of this is $-2^7 e^{2t}(\cos 2t + \sin 2t)$, i.e.,

$$\frac{\mathrm{d}^5(e^{2t}\sin(2t))}{\mathrm{d}t^5} = -2^7 e^{2t}(\cos 2t + \sin 2t).$$

5. What is the value of the integer n if $x^n - y^n$ is harmonic?

Solution: Recall that a function $u(x,y)$ is called harmonic if $u_{xx} + u_{yy} = 0$. So $x^n - y^n$ is harmonic if $n(n-1)(x^{n-2} - y^{n-2}) = 0$, which means that $n = 0$, $n = 1$, or $n = 2$.

6. Find the harmonic conjugate of $e^x \cos y + e^y \cos x + xy$.

Solution: With $u(x,y) = e^x \cos y + e^y \cos x + xy$, we need to find a function $v(x,y)$ such that $f = u + iv$ is analytic, that is, $u_x = v_y$ and $v_y = -u_y$. We have

$$u_x = e^x \cos y - e^y \sin x + y, \qquad u_y = -e^x \sin y + e^y \cos x + x.$$

Integrating u_x with respect to y, we get

$$v(x,y) = e^x \sin y - e^y \sin x + \tfrac{1}{2}y^2 + A(x),$$

where $A(x)$ is an arbitrary function of x. On the other hand, integrating $-u_y$ with respect to x, we have

$$v(x,y) = e^x \sin y - e^y \sin x + \tfrac{1}{2}x^2 + B(y).$$

where $B(y)$ is an arbitrary function of y. Combining the two expressions, we conclude that

$$v(x,y) = e^x \sin y - e^y \sin x + \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2,$$

satisfies $u_x = v_y$ and $v_y = -u_y$.

7. Compute $\arcsin(1+i)$.

   Solution: Recall the formula

$$\arcsin z = -i \log(zi \pm \sqrt{1-z^2}).$$

With $z = 1+i$, we get

$$\sqrt{1-z^2} = \sqrt{1-2i} = \sqrt[4]{5}\,(\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}),$$

where $\alpha = \arctan 2$, and so

$$w_{1,2} = zi \pm \sqrt{1-z^2} = -1 + i \pm \sqrt[4]{5}\,(\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}).$$

Now we recall

$$\log w = \operatorname{Log}|w| + i\arg w.$$

Note that $\arg w$ has infinitely many values differing from each other by integer multiples of 2π. Hence each of w_1 and w_2 results in an infinite collection of values for arcsin. We cannot do much beyond this, except to say that we can compute an approximate value by a calculator.

8. Prove that $\sin(iz) = i\sinh z$ and $\cos(iz) = \cosh z$.

   Solution: They follow from the definitions. For instance, the first identity is

$$\sin(iz) = \frac{e^{i\cdot iz} - e^{-i\cdot iz}}{2i} = \frac{e^{-z} - e^{z}}{2i} = i\frac{e^{z} - e^{-z}}{2} = i\sinh z.$$

9. Find all solutions of $\sin z - \cos z = 0$.

   Solution: The equation says that

$$\frac{e^{iz} - e^{-iz}}{2i} - \frac{e^{iz} + e^{-iz}}{2} = \frac{(1-i)e^{iz} - (1+i)e^{-iz}}{2i} = 0.$$

Multiplying it by $2ie^{iz}$, we arrive at

$$(1-i)e^{2iz} - (1+i) = 0, \qquad \text{or} \qquad e^{2iz} = \frac{1+i}{1-i} = i.$$

This implies that $2iz = i(\frac{\pi}{2} + 2\pi k)$ for some integer k, i.e.,

$$z = \frac{\pi}{4} + \pi k.$$

10. Compute $\log e^{e^i}$.

Solution: We have

$$z = e^{e^i} = e^{\cos 1 + i \sin 1} = e^{\cos 1}(\cos\sin 1 + i \sin\sin 1),$$

hence $|z| = e^{\cos 1}$ and $\arg z = \sin 1 + 2\pi k$, with integers k. We conclude

$$\log z = \operatorname{Log}|z| + i \arg z = \cos 1 + i(\sin 1 + 2\pi k).$$

11. Find all solutions of $e^z = e^{iz}$.

Solution: The exponential function has the period $2\pi i$, so $z = iz + 2\pi i k$ for some integer k. In other words,

$$z = \frac{2\pi i k}{1-i} = (1+i)\pi i k = (-1+i)\pi k.$$

12. Compute π^i.

Solution: We compute

$$\pi^i = e^{i\log\pi} = e^{i(\operatorname{Log}\pi + i\cdot 2\pi k)} = e^{i\operatorname{Log}\pi - 2\pi k} = e^{-2\pi k}(\cos\operatorname{Log}\pi + i\sin\operatorname{Log}\pi).$$

13. Find all solutions of $\sin\cos z = 0$.

Solution: Solving the outer equation gives

$$\cos z = \pi k,$$

for some integer k. Then taking the arccosine of it, we have

$$z = \arccos(\pi k) = -i\log(\pi k \pm \sqrt{\pi^2k^2 - 1}).$$

For $k \neq 0$, we have $\pi^2 k^2 > 1$, so

$$z = -i\log(\pi k \pm \sqrt{\pi^2k^2 - 1}) = -i\operatorname{Log}(\pi k \pm \sqrt{\pi^2k^2 - 1}) + 2\pi n,$$

for some integer n. The case $k = 0$, on the other hand, leads to

$$z = -i\log(\pm i) = -i \cdot i(\frac{\pi}{2} + 2\pi n) = \frac{\pi}{2} + 2\pi n,$$

for some integer n.

14. Evaluate $\int e^z \mathrm{d}z$, from $z = 1$ to $z = 1 + i$ along the line $x = 1$.

Solution: Let us parameterize the line by $z(t) = 1 + it$, with $0 \leq t \leq 1$. By definition of the complex line integral, we have

$$\int e^z \mathrm{d}z = \int_0^1 e^{z(t)} z'(t)\mathrm{d}t = \int_0^1 e^{1+it} i\mathrm{d}t = e^{1+it}\Big|_0^1 = e(e^i - 1).$$

15. Evaluate $\int \frac{\mathrm{d}z}{z}$, from $-i$ to i along the arc given by $z(t) = e^{it}$ with $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

Solution: Again by definition, we have

$$\int \frac{\mathrm{d}z}{z} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{z'(t)\mathrm{d}t}{z(t)} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ie^{it}\mathrm{d}t}{e^{it}} = it\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi i.$$