Homework 1 Solutions

Problem 1

$$\left(\frac{4-4i}{2+2i}\right)^7 + \left(\frac{4+4i}{2-2i}\right)^7$$

The second term is the conjugate of the first thus it is equal to $2\,\mathrm{Re}\left(\frac{4-4i}{2+2i}\right)^7$

$$\left(\frac{4-4i}{2+2i}\right)^7 = 2^7\left(\frac{1-i}{1+i}\right)^7 = \frac{2^7}{2^7}((1-i)^2)^7 = (-2i)^7 = (-1)^7 2^7 i^7 = 128i$$

Therefore we have the equation equal to $2\,\mathrm{Re}\left(\frac{4-4i}{2+2i}\right)^7 = 2\,\mathrm{Re}(128i) = 0$

Problem 2

This is clear since $\overline{z\bar{z}z} = \bar{z}\bar{\bar{z}}\bar{z} = \bar{z}z\bar{z}$

Problem 3

We have $z_1 = \alpha + i\beta, z_1 = \alpha - i\beta$, so $|z_1 + z_2| = 2|\alpha| = a$. Try $\alpha = a/2$

$$|z_1| + |z_2| = \sqrt{\alpha^2+\beta^2} + \sqrt{\alpha^2+\beta^2} = \frac{1}{a} \qquad 2\sqrt{\alpha^2+\beta^2} = \frac{1}{a} \;\; so \;\; \alpha^2+\beta^2 = \frac{1}{4a^2}$$

substituting back α we get $\beta = \pm\frac{1}{2}\sqrt{\frac{1}{a^2} - a^2}$ which gives us

$$z_1 = \frac{a}{2} + \frac{i}{2}\sqrt{\frac{1}{a^2} - a^2} \quad and \quad z_2 = \frac{a}{2} - \frac{i}{2}\sqrt{\frac{1}{a^2} - a^2}$$

Problem 4

$-\sqrt{3} + i = 2e^{i\frac{5\pi}{6}}$ so we have $(-\sqrt{3}+i)^{-\frac{1}{5}} = \frac{1}{\sqrt[5]{2}}e^{i(\frac{-\pi}{6} - \frac{2k\pi}{5})}$ for $k = 0,1,2,3,4$ equivalently

$$(-\sqrt{3}+i)^{-\frac{1}{5}} = \frac{1}{\sqrt[5]{2}}\left(\cos\left(\frac{5\pi + 12k\pi}{30}\right) - i\sin\left(\frac{5\pi+12k\pi}{30}\right)\right) \quad for \quad k = 0,1,2,3,4$$

Problem 5

a) We can extract the term

$$\begin{aligned}(z-1)(z^n + z^{n-1} + \cdots + z + 1) &= z(z^n + z^{n-1} + \cdots + z + 1) - 1(z^n + z^{n-1} + \cdots + z + 1) \\ &= zz^n + (zz^{n-1} + zz^{n-1} + z) - (z^n + z^{n-1} + \cdots + z) - 1 \\ &= z^{n+1} - 1\end{aligned}$$

b) Use the previous result

$$z^4 + z^3 + z^2 + z + 1 = \frac{z^5 - 1}{z - 1} \quad for \quad z \neq 1$$

So the roots of above is roots of $z^5 - 1 = 0$ so the answer is $z = cis(\frac{2\pi k}{5})$ for $k = 1, 2, 3, 4$. Notice there is one root missing, it is because $z \neq 1$ so $k = 0$ is not allowed.

Problem 6

a) Lets compute both expressions

$$(z^n)^{\frac{1}{m}} = r^{\frac{n}{m}} cis\left(\frac{n\theta}{m} + \frac{2\pi k}{m}\right) = r^{\frac{n}{m}} cis\left(\frac{n\theta}{m}\right) cis\left(\frac{2\pi k}{m}\right) \quad for \quad k = 0, 1, \cdots, m-1$$

$$(z^{\frac{1}{m}})^n = r^{\frac{n}{m}} cis\left(\frac{n\theta}{m} + \frac{2\pi nk}{m}\right) = r^{\frac{n}{m}} cis\left(\frac{n\theta}{m}\right) cis\left(\frac{2\pi nk}{m}\right) \quad for \quad k = 0, 1, \cdots, m-1$$

Both set of values are identical only when for each $cis\left(\frac{2\pi k_1}{m}\right)$ we can find k_2 such that it is equal to $cis\left(\frac{2\pi nk_2}{m}\right)$ for $k_1, k_2 \in \{0, 1, \cdots, m-1\}$.This is true since $\gcd(m, n) = 1$ so there exist $a, b \in \mathbb{Z}$ such that $am + bn = 1$, thus $bn = 1 \mod (m)$ where we can replace b by k_2 since $b = Nm + k_2$ for some $N \in \mathbb{Z}$. Therefore we have $nk_2 = 1 \mod (m)$ so by multiplying each side with k_1 we are able to get $nk_2 = k_1 \mod (m)$. Therefore for any k_1 we can find nk_2 such that they are equal up to $\mod (m)$ which implies $cis\left(\frac{2\pi k_1}{m}\right) = cis\left(\frac{2\pi nk_2}{m}\right)$ as we needed.

b)

$$(1^{\frac{1}{4}})^2 = \left[cis\left(\frac{2\pi k}{4}\right)\right]^2 = \pm 1 \quad for \quad k = 0, 1, 2, 3$$
$$(1^2)^{\frac{1}{4}} = 1^{\frac{1}{4}} = \left[cis\left(\frac{2\pi k}{4}\right)\right] = \pm 1, \pm i \quad for \quad k = 0, 1, 2, 3$$

Problem 7

It is a circle of radius half centered at $(\frac{1}{2}, 0)$, since

$$x^2 + y^2 = x \quad x^2 - x + y^2 = 0 \quad (x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$$

Problem 8

$$x + iy + \frac{1}{x + iy} = x + iy + \frac{x - iy}{x^2 + y^2}$$

Thus

$$U = x + \frac{x}{x^2+y^2} \quad , \quad V = y - \frac{y}{x^2+y^2}$$

Problem 9

We can write it as

$$x + iy + \frac{x+iy}{x^2+y^2} = z + \frac{z}{z\bar{z}} = z + \frac{1}{\bar{z}}$$

Problem 10

The function $f(z) = z$ is continuous everywhere, so $x = \text{Re}$ and $y = \text{Im}$ continuous everywhere. The sum of two continuous function is continuous thus $f(z) = \bar{z}$ is continuous everywhere.

The function $\frac{z-i}{\bar{z}-i}$ us quotient of two continuous functions, therefore it is continuous everywhere except when the denominator is zero, in particular when $\bar{z} - i = 0, \bar{z} = i \;\; so \;\; z = -i$.

Problem 11

We can take the limit from three different lines segments as follows

$$\lim_{y=0,x>0} f(z) = \frac{\sin(x)}{x} \to 1$$

$$\lim_{y>0,x=0} f(z) = \frac{i\sin(y)}{-iy} = -\frac{\sin(y)}{y} \to -1$$

$$\lim_{y=0,x=0} f(z) = \frac{(1+i)\sin(x)}{(1-i)x} = i\frac{\sin(x)}{x} \to i$$

Thus the limit does not exist.

Problem 12

Consider $f(z) = \frac{1}{z}$ which does not have limit as $z \to 0$ and $g(z) = -\frac{1}{z}$ also does not have limit as $z \to 0$. But $h(z) = f(z) + g(z) = 0$ has limit everywhere inside the complex plane.

Problem 13

$$\frac{d}{dz} z^{-5} = -5z^{-6}$$

This is defined on everywhere except $z = 0$.

Problem 14

We are given $U = y - xy, V = -x + x^2 - y^2$ use CR equations to check differentiability.

$$\frac{\partial U}{\partial x} = -2y \quad \frac{\partial V}{\partial x} = -2y \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \forall x, y$$

$$\frac{\partial U}{\partial x} = 1 - 2x \quad \frac{\partial V}{\partial x} = -(1 - 2x) \quad \frac{\partial U}{\partial x} = -\frac{\partial V}{\partial y} \quad \forall x, y$$

Therefore derivative exist everywhere since CR equation is satisfied.

Problem 15

a)

$z^3 + z^2 + 1$ is a sum of entire functions, thus it is analytic everywhere.

b)

$f'(z)|_{1+i} = 3z^2 + 2z = 3(1+i)^2 + 2 + 2i = 2 + 8i$ as needed.

Problem 16

a)

Since z^2 has derivative everywhere we need to consider only the second half. Thus we have $U = (x-1)^2 \;\; V = (y-1)^2$ and we can check CR equations

$$\frac{\partial U}{\partial x} = 2(x-1) \quad \frac{\partial V}{\partial y} = 2(y-1)$$

$$\frac{\partial U}{\partial x} = 0 \quad \frac{\partial V}{\partial y} = 0 \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \quad \forall x, y$$

Thus the function has derivative only when $2(x-1) = 2(y-1)$ which is equivalent to $x = y$. Therefore $f(z) = z^2 + (x-1)^2 + (y-1)^2$ have derivative only along the line x=y.

b)

The function is nowhere analytic since any neighborhood of z_0 on the line contain points z where $f'(z)$ is not defined.

c)

$$f'(z) = 2z + \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = 2z + 2(x-1) = 2 + 2i \quad at \quad (1,1)$$

Problem 17

a)

We are given that $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$ thus any path fixing x axis and any path fixing y axis has derivative 0 thus must be constant. Any point on the domain can be reached by sequence of paths such that on each of them it fixes one of the axis (staircase path). Thus every point must be same as any other points since they are connected by staircase path. Thus the function is constant on the domain.

b)

If the function is purely real on the domain then f can be written as $f(z) = U + i0$ we also know that CR equation is satisfied thus $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = 0$ and $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} = 0$, by part a) we see that function is constant on the domain. If the function is imaginary on the domain then $f(z) = 0 + iV$ so also by the same reasoning as above the function is constant on the domain.