SOLUTIONS TO SELECTED PROBLEMS FROM ASSIGNMENT 5

MATH 319 WINTER 2016

Problem 9.1:14

Statement. The speed a of wave propagation in a rectangular drum is 600 ft./sec.. The lowest two frequencies of the drum are 300 and 400 cycles per second. What are the length and the width of the drum?

Solution. We have the formua

$$\nu_{n,m} = \frac{a}{2}\sqrt{\frac{n^2}{L^2} + \frac{m^2}{M^2}},$$

for the frequency of the (n, m)-th harmonic, cf. Equation (33) in §9.1 of the textbook. Without loss of generality, we can assume $M \leq L$. Then the lowest two frequencies are $\nu_{1,1}$ and $\nu_{2,1}$, and therefore we have the equations

$$\nu_{1,1}^2 = \frac{a^2}{4}\Big(\frac{1}{L^2} + \frac{1}{M^2}\Big), \qquad \nu_{2,1}^2 = \frac{a^2}{4}\Big(\frac{4}{L^2} + \frac{1}{M^2}\Big).$$

We know a, $\nu_{1,1}$, and $\nu_{2,1}$, so it should be possible to find L and M from these two equations. Rearranging the equations, we infer

$$\begin{aligned} \frac{1}{L^2} + \frac{1}{M^2} &= \frac{4\nu_{1,1}^2}{a^2} = 1\,\text{ft.}^{-2}, \\ \frac{4}{L^2} + \frac{1}{M^2} &= \frac{4\nu_{2,1}^2}{a^2} = \frac{16}{9}\,\text{ft.}^{-2}, \end{aligned}$$

which can easily be solved as

$$\frac{1}{L^2} = \frac{7}{27}\,\text{ft.}^{-2}, \qquad \frac{1}{M^2} = \frac{20}{27}\,\text{ft.}^{-2}.$$

Thus, the length and the width of the drum are

$$L = \frac{3\sqrt{21}}{7}\,\text{ft.} \approx 1.96\,\text{ft.}, \qquad M = \frac{3\sqrt{15}}{10}\,\text{ft.} \approx 1.16\,\text{ft.}.$$

Problem B1

Statement. For each of the following situations in the xy-plane, compute the monopole, dipole, and quadrupole moments, and write down the multipole expansion of the electrostatic potential up to (and including) the quadrupole term. Normalize the constant so that the potential of a unit charge at the origin would be $\log \frac{1}{r}$, where r is the radial coordinate.

(a) The rectangle $R = \{(x, y) : -1 \leq x \leq 1,\ -h \leq y \leq h\}$ with total charge 1, and uniform charge density. Here $h > 0$ is a given constant.

(b) The same as in (a), but now the half $R \cap \{x > 0\}$ of the rectangle has total charge $+1$ with uniform density, and the other half has total charge -1 with uniform density.

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Solution. (a) Since the total charge is 1, the monopole moment is $a_0 = 1$. The area of the rectangle is $|R| = 4h$, so the charge density is $\mu = \frac{1}{4h}$. We compute the dipole moment as

$$a_1 = \int_{-1}^{1}\int_{-h}^{h} x\mu \,\mathrm{d}y\mathrm{d}x = 0, \qquad b_1 = \int_{-1}^{1}\int_{-h}^{h} y\mu \,\mathrm{d}y\mathrm{d}x = 0.$$

As for the quadrupole moment, we have

$$\begin{aligned} a_2 &= \frac{1}{2}\int_{-1}^{1}\int_{-h}^{h}(x^2-y^2)\mu \,\mathrm{d}y\mathrm{d}x = \frac{1}{2}\cdot 2h\mu\int_{-1}^{1} x^2\mathrm{d}x - \frac{1}{2}\cdot\frac{2h^3}{3}\mu\int_{-1}^{1}\mathrm{d}x \\ &= \frac{2}{3}h\mu - \frac{2}{3}h^3\mu = \frac{1}{6} - \frac{1}{6}h^2, \end{aligned}$$

and

$$b_2 = \int_{-1}^{1}\int_{-h}^{h} xy\mu \,\mathrm{d}y\mathrm{d}x = 0.$$

Hence the potential generated by the charged rectangle, up to the quadrupole term is

$$u(r,\theta) = \log\frac{1}{r} + \frac{(1-h^2)\cos 2\theta}{r^2} + \ldots.$$

(b) In this case, the total charge is 0, and the charge density is $\mu(x) = \pm\frac{1}{2h}$ depending on whether $x > 0$ or $x < 0$. Obviously, there is no monopole moment: $a_0 = 0$. For the dipole moment, we have

$$\begin{aligned} a_1 &= \frac{1}{2h}\int_{0}^{1}\int_{-h}^{h} x \,\mathrm{d}y\mathrm{d}x - \frac{1}{2h}\int_{-1}^{0}\int_{-h}^{h} x \,\mathrm{d}y\mathrm{d}x = \frac{1}{2h}\cdot 2h\int_{0}^{1} x\,\mathrm{d}x - \frac{1}{2h}\cdot 2h\int_{-1}^{0} x\,\mathrm{d}x \\ &= \frac{1}{2} - \big(-\frac{1}{2}\big) = \frac{1}{4}, \end{aligned}$$

and

$$b_1 = \frac{1}{2h}\int_{0}^{1}\int_{-h}^{h} y \,\mathrm{d}y\mathrm{d}x - \frac{1}{2h}\int_{-1}^{0}\int_{-h}^{h} y \,\mathrm{d}y\mathrm{d}x = 0 - 0 = 0,$$

because y is an odd function of y. Next, we compute the quadrupole moment as

$$\begin{aligned} a_2 &= \frac{1}{2}\cdot\frac{1}{2h}\int_{0}^{1}\int_{-h}^{h}(x^2-y^2)\,\mathrm{d}y\mathrm{d}x - \frac{1}{2}\cdot\frac{1}{2h}\int_{-1}^{0}\int_{-h}^{h}(x^2-y^2)\,\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{2}\cdot\frac{1}{2h}\int_{0}^{1}\int_{-h}^{h}(x^2-y^2)\,\mathrm{d}y\mathrm{d}x - \frac{1}{2}\cdot\frac{1}{2h}\int_{0}^{1}\int_{-h}^{h}(x^2-y^2)\,\mathrm{d}y\mathrm{d}x = 0, \end{aligned}$$

because $x^2 - y^2$ is an even function of x, and

$$b_2 = \frac{1}{2h}\int_{0}^{1}\int_{-h}^{h} xy \,\mathrm{d}y\mathrm{d}x - \frac{1}{2h}\int_{-1}^{0}\int_{-h}^{h} xy \,\mathrm{d}y\mathrm{d}x = 0 - 0 = 0,$$

because xy is an odd function of y. The conclusion is that

$$u(r,\theta) = \frac{\cos\theta}{4r} + O(r^{-3}),$$

where the error term $O(r^{-3})$ is there to indicate that the quadrupole term is in fact 0.

Problem B2

Statement. Let $u(x, y, t)$ be a smooth function satisfying

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} & \text{in } \Omega, \quad \text{for } t > 0, \\ u = 0 & \text{on } \partial\Omega, \quad \text{for } t > 0, \\ u(x, y, 0) = f(x, y) & \text{for } (x, y) \in \Omega, \\ u_t(x, y, 0) = g(x, y) & \text{for } (x, y) \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded region in the plane, and f and g are given initial data. In other words, u is a solution of the Dirichlet initial-boundary value problem for the wave equation in the domain Ω. By using the energy method, show that u is the unique solution of this problem, i.e., that there are no other solutions. *Hint*: Suppose that $v(x, y, t)$ is another solution of the problem (with the same initial and boundary data), and consider the energy

$$E(t) = \int_\Omega (w_t^2 + w_x^2 + w_y^2), \tag{1}$$

for difference $w = u - v$. Compute the time derivative $E'(t)$. Apply the divergence theorem to the vector field $F = w_t \operatorname{grad} w$.

Solution. As suggested in the hint, let $v(x, y, t)$ be another solution of the problem with the same initial and boundary data, and let $w(x, y, t) = u(x, y, t) - v(x, y, t)$. Then w satisfies

$$\begin{cases} w_{tt} = w_{xx} + w_{yy} & \text{in } \Omega, \quad \text{for } t > 0, \\ w = 0 & \text{on } \partial\Omega, \quad \text{for } t > 0, \\ w(x, y, 0) = 0 & \text{for } (x, y) \in \Omega, \\ w_t(x, y, 0) = 0 & \text{for } (x, y) \in \Omega. \end{cases}$$

For $E(t)$ defined by (1), we have

$$E'(t) = \int_\Omega (2w_t w_{tt} + 2w_x w_{xt} + 2w_y w_{yt}) = 2\int_\Omega (w_t \Delta w + w_x w_{xt} + w_y w_{yt}), \tag{2}$$

where we have used the wave equation $w_{tt} = \Delta w$. Now, for $F = w_t \operatorname{grad} w = (w_t w_x, w_t w_y)$, we compute

$$\operatorname{div} F = \frac{\partial}{\partial x}(w_t w_x) + \frac{\partial}{\partial y}(w_t w_y) = w_{tx} w_x + w_t w_{xx} + w_{ty} w_y + w_t w_{yy} = w_t \Delta w + w_x w_{xt} + w_y w_{yt},$$

which is exactly the expression under the integral in (2). Therefore, we infer

$$E'(t) = 2\int_\Omega \operatorname{div} F = 2\int_{\partial\Omega} n \cdot F = 2\int_{\partial\Omega} w_t (n \cdot \operatorname{grad} w) = 0, \tag{3}$$

where we have used the divergence theorem in the second step, the definition $F = w_t \operatorname{grad} w$ in the third step, and the fact that $w \equiv 0$ on $\partial\Omega$ in the last step. The equality (3) tells us that the energy $E(t)$ stays constant.

Since $w(x, y, 0) \equiv 0$, we have $w_x(x, y, 0) \equiv 0$ and $w_y(x, y, 0) \equiv 0$ as well, and hence

$$E(0) = \int_\Omega \left(w_t(x, y, 0)^2 + w_x(x, y, 0)^2 + w_y(x, y, 0)^2 \right) \mathrm{d}x\mathrm{d}y = 0.$$

By combining this result with (3), we conclude that $E(t) \equiv 0$ for all t. In view of (1), $E(t) \equiv 0$ implies that $w_t \equiv w_x \equiv w_y \equiv 0$. Then for an arbitrary point $(x, y) \in \Omega$ and an arbitrary time $t > 0$, we have

$$w(x, y, t) = w(x, y, 0) + \int_0^t w_t(x, y, s)\,\mathrm{d}s = 0 + 0 = 0.$$

This means that $u \equiv v$, that is, the solution u is unique.