Question

Given a rod with 10cm of length that has a constant heat flux of 2 and 3 at the left edge and the right edge respectively, restate the following question after finding an appropriate boundary conditions and ultimately find the U(x,t).

1. $U_x = 3U_{xx}\ , t \geq 0$
2. BC: Unknown
3. IC: $U(x,0) = \frac{1}{20}x^2 + 2x + \cos\frac{\pi}{x}$

Solution

Finding BC:

$$U_x(0,t) = 2, U_x(10,t) = 3$$

It is a Neumann condition with a constant heat flux at the boundaries.

Solution:

First we shift the unknown function so that the BC are homogeneous.

Let P(x) = Ax^2+$2x$ satisfying p'(0)=2 and p'(10)=3. Also A=$\frac{1}{20}$ and B=2.

Let let v=u-p so that u=p+v. Since U_t=V_t and U_{xx}=V_{xx}+2A, we see that v must satisfy

$$\begin{cases} v_t = 5v_{xx} + \frac{1}{2}, & 0 \leq x \leq 10,\ t \geq 0, \\ v_x(0,t) = 0, & v_x(10,t) = 0, \\ v(x,0) = u(x,0) - p(x) = \cos(\pi x). \end{cases}$$

Note that we have used the properties w_x(0,t)= v_x(0,t), w_x(10,t)= v_x(10,t), and w(x,0)=v(x,0). The problem for w can be solved easily, as

$$w(x,t) = e^{-5\pi^2 t}\cos(\pi x).$$

Which yields

$$u(x,t) = w(x,t) + \frac{1}{2}t + p(x) = \frac{1}{2}t + \frac{1}{20}x^2 + 2x + e^{-5\pi^2 t}\cos(\pi x).$$

Source: modification of textbook question

Reason: Understanding physical interpretation of dirichlet and neumann boundary conditions for heat equation in practice and apply appropriate boundary condition for solving a problem.

Problem Statement

Solve the Problem
$$\begin{cases} u_t - u_{xx} = 0 & -\pi \le x \le \pi \quad t \ge 0 \\ u(-\pi, t) = u(\pi, t) \\ u_x(-\pi, t) = u_x(\pi, t) \end{cases}$$

Given $f(x) = \frac{\pi}{2} + 3\sin(2x) - 5\cos(3x) - 6\sin(3x)$

Solution

$$u(x,t) = A_0 + \sum \left(a_n \cos(\lambda_n x) + b_n \sin(\lambda_n x)\right) e^{-\lambda_n^2 k t}$$

Periodic BC = sum of neumann & Dirichlet BC

$A_0 = \frac{\pi}{2}$ $\quad k = 1 \quad$ $\lambda_n = \frac{n\pi}{L} = \frac{n\pi}{\pi} = n$

$$\left[\begin{array}{ccc} -5\cos(3x) & +3\sin(2x) & -6\sin(3x) \\ n=3 & n=2 & n=3 \\ a_3 = -5 & b_2 = 3 & b_3 = -6 \end{array}\right]$$

$$\Rightarrow \boxed{u(x,t) = \frac{\pi}{2} - 5e^{-9t}\cos(3x) + 3e^{-4t}\sin(2x) - 6e^{-9t}\sin(3x)}$$

Explanation why well suited:

1. Student will have to recognize It is the heat equation
2. Student must identify we have periodic BC
3. Student will have to know that the solution is of the form: $u(x,t) = A_0 + \sum (a_n \cos(\lambda_n x) + b_n \sin(\lambda_n x))\, e^{-\lambda_n^2 k t}$
4. Student must identify the proper k, L, λ, a_n, b_n's to obtain The correct solution.

Sources: This problem is a variation of the exercises in the book by Bleecker. (Exercise 3.1-6) where the function $f(x)$ I created.

Problem statement: "The solution to the wave equation is bounded initially and remains bounded at all times"
Prove or provide a counter example.

Full solution:

The statement is false.
Counter-example:

Consider the following problem:

$$\begin{cases} u_{tt} = a u_{xx} & -\infty < x, t < \infty \\ u(x,0) = 1 \\ u_t(x,0) = 1 \end{cases}$$

The solution can be found using d'Alembert's formula since both $f(x)$ and $g(x)$ are C^2.

$$u(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds$$

$$= \frac{1}{2}\left[1 + 1\right] + \frac{1}{2a}\left[(x+at) - (x-at)\right]$$

$$= 1 + \frac{1}{2a}(2at)$$

$$\boxed{u(x,t) = 1 + t}$$

As $\lim_{t\to\infty}(1+t) = \infty$, we see that the solution is not bounded at all times.

Explanation: I think it is suited for a final exam because the solution doesn't have to be long, but it requires understanding of the wave equation both on an intuitive and mathematical perspective.

Source: www.maths.tcd.ie/~pete/pde/hs2.pdf #4

Q. Consider the solution $u(x,t)$ of the heat equation:

$$u_t = 4u_{xx} \qquad 0<x<2, \; t\geq 0$$
$$u_x(0,t) = u_x(2,t) = 0 \qquad , \; t>0$$
$$u(x,0) = g(x), \qquad 0<x<2.$$

Show that function $F(t) = -\int_0^2 e^{u(x,t)}\,dx$ is increasing for $t \geq 0$.

SOLUTION:

We compute $F'(t)$.

Then $F'(t) = \frac{d}{dt}\left[-\int_0^2 e^{u(x,t)}\,dx\right]$

$$= -\int_0^2 \left[\frac{d}{dt} e^{u(x,t)}\right] dx$$

$$= -\int_0^2 u_t(x,t)\, e^{u(x,t)}\, dx$$

$$= \int_0^2 4u_{xx}(x,t)\, e^{u(x,t)}\, dx \qquad \text{From the PDE}$$

$$= -4\left[\left\{u_x(x,t)e^{u(x,t)}\right\}\Big|_0^2 - \int_0^2 u_x(x,t)\frac{d}{dx}e^{u(x,t)}\,dx\right]$$

$$= -4\left[u_x(2,t)e^{u(2,t)} - u_x(0,t)e^{(0,t)} - \int_0^2 [u_x(x,t)]^2 e^{u(x,t)}\,dx\right]$$

after doing integration by parts

$$= \underbrace{-4u_x(2,t)e^{u(2,t)}}_{=0} \underbrace{+4u_x(0,t)e^{(0,t)}}_{=0} + \underbrace{4\int_0^2 [u_x(x,t)]^2 e^{u(x,t)}\,dx}_{\geq 0}$$

$\Rightarrow \quad F'(t) \geq 0$

Hence $F'(t) \geq 0$ for all $t>0$.

$\Rightarrow$ $F(t)$ is increasing for $t \geq 0$.

Dirichlet problem on Rectangle.

Q: Let $\Delta u = 0$, $0 < x < \pi$, $0 < y < 4$

$u(x,0) = 2\sin(4x)$

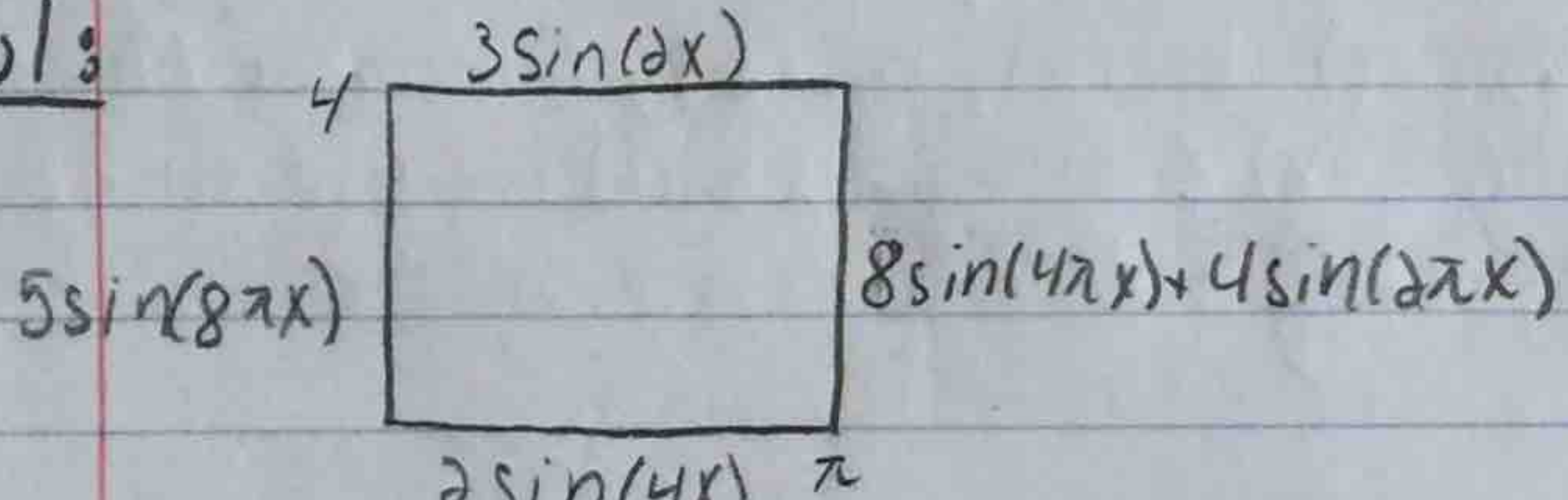
$u(x,4) = 2\sin(x)\cos(x) + 2\sin(2x)$

$u(0,y) = 5\sin(8\pi x)$

$u(\pi,y) = 8\sin(4\pi x) + 8\sin(\pi x)\cos(\pi x)$

Find $u(x,y)$.

Sol:



* Trigo Identity: $2\sin(a)\cos(a) = \sin(2a)$

$\Rightarrow u(x,4) = 2\sin(x)\cos(x) + 2\sin(2x)$

$= \sin(2x) + 2\sin(2x)$

$= 3\sin(2x)$

$\Rightarrow u(\pi,y) = 8\sin(4\pi x) + 4\sin(2\pi x)$

$a_4(0) = 2$, $a_1(0), a_3(0), a_5(0), \ldots = 0$ $\Rightarrow \lambda_4 = 4$

$a_2(4) = 3$, $a_1(4), a_3(4), \ldots = 0$ $\Rightarrow \lambda_2 = 2$

$b_8(0) = 5$, $b_1(0), \ldots b_7(0), b_9(0), \ldots = 0$. $\Rightarrow \mu_n = \frac{\pi n}{M} = \frac{8\pi}{4} = \boxed{2\pi = \mu_8}$

$b_2(\pi) = 4$

$b_4(\pi) = 8$ $\quad b_1(\pi), b_3(\pi), b_5(\pi) = 0$ $\Rightarrow \mu_2 = \frac{2\pi}{4} = \boxed{\pi/2}$ $\quad \mu_4 = \frac{4\pi}{4} = \boxed{\pi}$

$$u(x,y) = a_4(0)\frac{\sinh(\lambda_4(4-y))}{\sinh(\lambda_4 \cdot 4)}\sin(\lambda_4 x) + a_2(4)\frac{\sinh(\lambda_2 y)}{\sinh(\lambda_2 \cdot 4)}\sin(\lambda_2 x)$$

$$+ b_8(0)\frac{\sinh(\mu_8(\pi - x))}{\sinh(\mu_8 \cdot \pi)}\sin(\mu_8 y) + b_2(\pi)\frac{\sinh(\mu_2 x)}{\sinh(\mu_2 \cdot \pi)} \cdot \sin(\mu_2 y)$$

$$+ b_4(\pi)\frac{\sinh(\mu_4 x)}{\sinh(\mu_4 \pi)}\sin(\mu_4 y)$$

$$u(x,y) = 2\frac{\sinh(16-4y)}{\sinh(16)}\sin(4x) + 3\frac{\sinh(2y)}{\sinh(8)}\sin(2x).$$

$$+ 5\frac{\sinh(2\pi^2 - 2\pi x)}{\sinh(2\pi^2)}\sin(2\pi y) + \frac{4\sinh(\pi x/2)}{\sinh(\pi^2/2)}\sin(\pi y/2)$$

$$+ 8\frac{\sinh(\pi^2 - \pi x)}{\sinh(\pi^2)}\sin(\pi y)$$

This problem require the students to know how to use the formulas for $u(x,y)$; what is λ_n, how to identify the coefficients $a_n(0)$, $a_n(M)$, $b_n(0)$, $b_n(L)$, remember that for $a_n(0)$ & $b_n(0)$ we use $\sinh(\_\_ - x)$ & for $a_n(M)$ & $b_n(L)$ we use $\sinh(\_\_ \cdot x)$. Furthermore, since $M=4$, they'll need to remember that $\mu_n = \frac{\pi n}{M}$.

The initial problem also contains cosine, which we haven't seen for this type of problem, they'll thus need to chang it back into sine fct.

It requires a bit of thinking but stays doable. If you know how to do it, it should be doable in less than 5 minutes but if you have not practiced enough, you won't be able to do it.

STATEMENT OF THE PROBLEM:

Solve:

D.E: $u_{tt} = u_{xx}$ $0<x<\infty$, $0<t<\infty$

B.C: $u_x(0,t) = 0$

I.C: $u(x,0) = \frac{x^2}{e^x}$ $u_t(x,0) = 0$

FULL SOLUTION

. We are going to use D'Alembert's method to solve the above problem. Firstly we need to extend the initial data by even reflection to the entire real line, then apply D'Alembert's formula and then restrict it to the half line $x \geq 0$. The even reflection of $\frac{x^2}{e^x}$ is $f(x) = \frac{x^2}{e^{|x|}}$.

The general solution then is given by: $u(x,t) = F(x-t) + G(x+t)$ and from I.C. we have $\longrightarrow \begin{cases} F(x) + G(x) = f(x) \\ G'(x) - F'(x) = 0 \end{cases}$

Therefore $F(x) = G(x) = \frac{f(x)}{2}$ and therefore we have for our solution

$$u(x,t) = \frac{1}{2} \frac{(x+t)^2}{e^{|x+t|}} + \frac{1}{2} \frac{(x-t)^2}{e^{|x-t|}}$$

EXPLANATION

. (i) The problem is taken from the chapter 5.2 exercise 4 and modified. The original problem is:

D.E: $u_{tt} = a^2 u_{xx}$ $0 \leq x \leq \infty$, $-\infty < t < \infty$

B.C: $u_x(0,t) = 0$

I.C: $u(x,0) = x^3$, $u_t(x,0) = 0$

I have set $a = 1$ and instead of x^3 I changed it to x^2/e^x. Making the problem somewhat harder looking while maintaining it's structure.

<u>Problem</u>

Solve the following problem:

$$u_t - u_{xx} = t\sin x + \frac{x}{\pi}, \quad 0 \le x \le \pi, \quad t \ge 0$$
$$u(0,t) = 0, \quad u(\pi,t) = t$$
$$u(x,0) = \sin x$$

<u>Solution</u>

We begin by finding a particular solution $w(x,t)$ that satisfies the boundary conditions. Guess $w(x,t) = a(t)x + b(t)$, then:

$$w(0,t) = b(t) = 0$$
$$w(\pi,t) = a(t)\pi = t \Rightarrow a(t) = \frac{t}{\pi}$$

$$\Rightarrow w(x,t) = \frac{xt}{\pi}$$

Now let $u(x,t) = v(x,t) + w(x,t)$, and consider the related problem for v: $(v = u - w)$

$$v_t - v_{xx} = t\sin x + \frac{x}{\pi} - \left(\frac{x}{\pi}\right)$$
$$= t\sin x$$
$$v(0,t) = 0, \quad v(\pi,t) = 0$$
$$v(x,0) = \sin(x)$$

We now decompose this problem into the following two problems for u_1 and u_2, with $v(x,t) = u_1(x,t) + u_2(x,t)$.

$$(u_1)_t - (u_1)_{xx} = 0$$
$$u_1(0,t) = 0, \quad u_1(\pi,t) = 0$$
$$u_1(x,0) = \sin x$$

$$(u_2)_t - (u_2)_{xx} = t\sin(x)$$
$$u_2(0,t) = 0, \quad u_2(\pi,t) = 0$$
$$u_2(x,0) = 0$$

We write down the solution for u_1 using Proposition 1 of §3.1.

$$u_1(x,t) = e^{-t}\sin x$$

To solve for u_2, we apply Duhamel's Principle. Consider the following related problem:

$$\tilde{v}_t - \tilde{v}_{xx} = 0$$
$$\tilde{v}(0,t;s)\ 0, \quad \tilde{v}(\pi,t;s)\ 0$$
$$\tilde{v}(x,0;s) = s\sin(x)$$

Then $$u_2(x,t) = \int_0^t \tilde{v}(x,t-s;s)\,ds.$$

From Proposition 1, the solution of $\tilde{v}$ is:

$$\tilde{v}(x,t;s) = s e^{-t}\sin x$$

Thus: $$u_2(x,t) = \int_0^t s e^{-(t-s)}\sin x\,ds$$
$$= e^{-t}\sin x \int_0^t s e^{s}\,ds$$

$$= e^{-t} \sin x \left[e^{x} (x-1) \right]_0^t$$

$$= e^{-t} \sin x \left[e^{t} (t-1) + 1 \right]$$

$$= \sin x (t-1) + e^{-t} \sin x$$

The full Solution is thus:

$$u(x,t) = w(x,t) + u_1(x,t) + u_2(x,t)$$

$$= \frac{xt}{\pi} + \sin(x)\left(2e^{-t} + t - 1 \right)$$

<u>Check</u>:

$$u_t = \frac{x}{\pi} + \sin(x)\left(-2e^{-t} + 1 \right)$$

$$-u_{xx} = -\left(-\sin(x)\left(2e^{-t} + t - 1\right)\right)$$

$$\Rightarrow \quad u_t - u_{xx} = \frac{x}{\pi} + t \sin t \quad \checkmark$$

$$u(0,t) = 0 \checkmark$$

$$u(\pi,t) = t \checkmark$$

$$u(x,0) = \sin(x)(2-1) = \sin x \checkmark$$

So this is indeed the Solution.

(pg. [illegible] of text helped)

The problem I am proposing is a Dirichlet problem with inhomogeneous boundary conditions. I choose this problem because it has multiple steps because it is inhomogeneous but also allows students to check their solution. I used the text and class notes to help me write the problem and solve it, however it is not a direct modification of any existing problem I found.

Problem:

D.E. $u_t - u_{xx} = 0 \quad 0 < x < \pi, \quad t \geq 0$

I.C. $u(0,t) = 3t, \ u(\pi,t) = 7t$

B.C. $u(x,0) = 7\sin(2x)$

We want a function of form $w(x,t) = c(t)x + d(t)$ satisfies the boundary conditions:

$w(0,t) = c(t)\cdot 0 + d(t) = 3t \Rightarrow d(t) = 3t$

$w(\pi,t) = c(t)\cdot \pi + 3t = 7t \Rightarrow c(t) = \frac{4t}{\pi}$

$w(x,t) = \frac{4t}{\pi}x + 3t$

$w_t(x,t) = \frac{4x}{\pi} + 3 \quad w_x(x,t) = \frac{4t}{\pi} + 0 \quad w_{xx} = 0$

$\therefore w_t - w_{xx} = \frac{4x}{\pi} + 3 \quad w(x,0) = 0$

We want a solution of the form $u(x,t) = w(x,t) + v(x,t)$ where $v(x,t)$ solves the related problem:

1

 $u_t - u_{xx} = 0$ $\quad w_t - w_{xx}$

D.E. $V_t - V_{xx} = 0 - \left[\frac{4x}{\pi} + 3\right] = \frac{-4x}{\pi} - 3 \qquad 0 \le x \le \pi \quad t \ge 0$

I.C. $V(0,t) = 3t - 3t = 0 \qquad V(\pi,t) = 7t - 7t = 0$

B.C. $V(x,0) = 7\sin(2x) - 0 = 7\sin(2x)$

The related problem has inhomogeneous D.E. with homogeneous boundary conditions and the solution is split into two further problems: $V(x,t) = u_1(x,t) + u_2(x,t)$

D.E. $(u_1)_t - (u_1)_{xx} = 0 \qquad (u_2)_t - u_{2\,xx} = -\frac{4x}{\pi} - 3$

$u_1(0,t) = u_1(L,t) = 0 \qquad u_2(0,t) = u_2(\pi,t) = 0$

$u_1(x,0) = 7\sin(2x) \qquad u_2(x,0) = 0$

The solution to $u_1(x,t)$ is of the form:

$u_1(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda n^2 t} \sin(\lambda n) \qquad \lambda_n = \frac{n\pi}{L} = n$

$\Rightarrow u_1(x,t) = 7e^{-4t}\sin(2x)$

To solve $u_2(x,t)$, we solve the related problem:

D.E. $q_t - q_{xx} = 0$

B.C. $q(0,t) = q(\pi,t) = 0$

I.C. $q(x,0) = h(x,s) = \frac{-4s}{\pi} - 3$

$u_2(x,t) = \int_0^t \left(-\frac{4s}{\pi} - 3\right) e^{-k(t-s)}$

$= e^{-kt} \int^t \left(-\frac{4s}{\pi} - 3\right) e^{ks}\, ds \qquad (k=1)$

$= e^{-t} \left[e^s \left(-\frac{4s}{\pi} - 3s\right) \right]_0^t$

$= e^{-t} \left[e^t \left(-\frac{4t^2}{\pi} - 3t\right) - 1 \cdot (0) \right]$

$= -\frac{4t^2}{\pi x} - 3t$

$\therefore\ u_2(x,t) = \frac{-4t}{\pi}x - 3t$

So, $v(x,t) = u_1(x,t) + u_2(x,t)$
$= 7e^{-4t}\sin(2x) - \frac{4t}{\pi}x - 3t$

and $u(x,t) = v(x,t) + w(x,t)$
$= 7e^{-4t}\sin(2x) \cancel{- \frac{4t}{\pi}x} \cancel{- 3t} \cancel{+ \frac{4t}{\pi}x} \cancel{+ 3t}$
$= 7e^{-4t}\sin(2x)$

<u>Check:</u> $u_t = -28e^{-4t}\sin(2x)$

$u_x = 14e^{-4t}\cos(2x)$

$u_{xx} = -28e^{-4t}\sin(2x)$

$u_t - u_{xx} = -28e^{-4t}\sin(2x) - (-28e^{-4t}\sin(2x)) = 0$ ✓

Problem Statement

Consider the following heat problem:

$$\begin{aligned} &\text{DE} \quad u_t = K u_{xx}, \quad 0 \leq x \leq 10, \; t \geq 0 \\ &\text{BC} \quad u_x(0,t) = 0, \quad u_x(1,t) = 0. \\ &\text{IC} \quad u(x,0) = F(x). \end{aligned}$$

Knowing that the solution to this problem is

$u(x,t) = -1 + 18\cos(3\pi x/2)\, e^{-9\pi^2 t}$, find the corresponding

initial condition of this problem, $F(x)$, and the value of K.

Full solution

It is clear that we have:

- Heat problem with $K = 4$, $L = 10$.
- Homogeneous Neumann Boundary Conditions.
- Unknown initial condition!

In general, the solution to such a problem has the following form:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda_n x)\, e^{-\lambda_n^2 K t}$$

where A_i are coefficients from

$$F(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda_n x)$$

(continued)

Finding F(x)

$u(x,t) = -1 + 18\cos(3\pi x/2)\, e^{-9\pi^2 t}$

$\therefore$ Knowing the FCS expansion of F(x),

$A_0 = -1$

$A_m = 18 \longrightarrow$ need to find m.

$\lambda_m x = 3\pi x/2$

$\cancel{x}\, m\cancel{\pi}/L = 3\cancel{\pi}\cancel{x}/2$

$m/10 = 3/2$

$\therefore m = 15$

so in F(x) we have $A_0 = -1$, $A_{15} = 18$ and $A_n = 0$ $\forall\, n \neq 0, 15$.

$\longrightarrow u(x,0) = F(x) = -1 + 18\cos(3\pi x/2)$

Finding K

We have the exponential term:

$e^{-9\pi^2 t}$ when $n = 15$

$-9\pi^2 \cancel{t} = -\lambda_n^2 K \cancel{t}$

$\cancel{-}\, 9\cancel{\pi}^2 = \cancel{-}\left(\frac{15\cancel{\pi}}{10}\right)^2 K$

$9 = \frac{9}{4}K \longrightarrow K = 4$

Explanation

This problem is good because:

- we have never done a problem where the solution is known but the IC is not (going backwards)
- student must recognize the form of both the solution of a homogeneous Neumann heat problem AND the Fourier cosine series expansion.

EXAM QUESTION SUGGESTION

Solve the heat equation:

$u_t = 2u_{xx} \qquad 0 \le x \le \pi \ , \ t \ge 0$

$u_x(0,t) = 0 \ , \ u_x(L,t) = 0$

$u(x,0) = x$

SOLUTION

From derivations in class and textbook, we know if $f(x)$ is of the form $\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

then the solution is of the form:

$$u(x,t) = \sum_{n=0}^{N} a_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \cos\left(\frac{n\pi x}{L}\right)$$

$\Rightarrow f(x) = x$ so can approximate its fourier cosine series since it is a continuous function

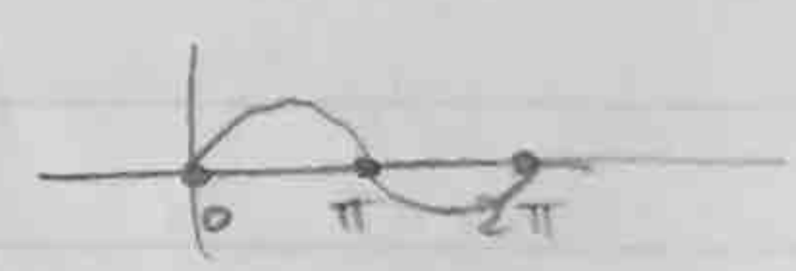
$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos(nx)\, dx$$

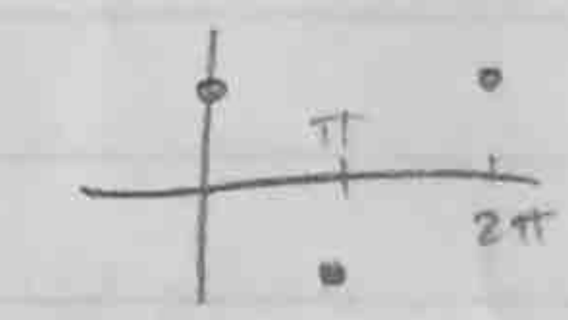
$\Rightarrow$ using integration by parts: $\quad u = x \quad du = dx$

$\qquad v = \frac{1}{n}\sin nx \quad dv = \cos nx$

$$a_n = \frac{2}{\pi}\left[\frac{x}{n}\sin nx \Big|_0^{\pi} - \int_0^{\pi} \frac{1}{n} \sin nx \, dx \right]$$



$$= \frac{2}{\pi}\left[\left(\frac{\pi}{n}\sin(n\pi) - 0\right) - \left[-\frac{1}{n^2}\cos nx\right]\Big|_0^{\pi}\right]$$



$$= \frac{2}{\pi}\left[\underbrace{\frac{\pi}{n}\sin(n\pi)}_{=0 \ \forall n \text{ integers}} - \left[-\frac{1}{n^2}(\cos n\pi) + \frac{1}{n^2}\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

⇒ so cosine fourier series for $f(x)=x$ is:

$$x = \frac{2}{\pi}\sum_{n=0}^{\infty}\left(\frac{(-1)^n-1}{n^2}\right)\cos(nx)$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\sum_{n=0}^{N}\left(\frac{(-1)^n-1}{n^2}\right)e^{-n^2 2t}\cos(nx)$$

⇒ however, when n = even integer

$$a_n = \frac{(-1)^{2n}-1}{n^2} = \frac{1-1}{n^2} = 0$$

so a_n exists only for odd n

and when n = odd integer

$$a_n = \frac{(-1)^n-1}{n^2} = \frac{-1-1}{n^2} = \frac{-2}{n^2}$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\sum_{n\ \text{odd}}^{N}\left(\frac{-2}{n^2}\right)e^{-n^2 2t}\cos(nx)$$ is the final solution

EXPLANATION

I think this is a good problem for the final because it is relevant, as we have studied the heat equation extensively. As well, it is not completely trivial as one must first find the fourier cosine series for f(x), so it also touches on another topic that we have studied: fourier series. At the same time, the calculations are not too involved and tedious, which is good for a final when everyone is very stressed.

I modified this problem from chapter 3 in the text but then changed f(x) so that it also touched on chapter 4.

Example 1, page 157

problem: A rod has length $\ell = 1$ and constant $k = 1$. It's temperature satisfies the heat equation. Its left end is held at temperature 0 and its right end at temperature 1. Initially at time 0 the temperature is given by:

$$\phi(x) = \begin{cases} \frac{5x}{2} & \text{for } 0 < x < 2/3 \\ 3 - 2x & \text{for } 2/3 < x < 1. \end{cases}$$

Find the solution, including the coefficients.

Solution: we consider:

$$v(t,x) = u(t,x) - x$$

the function satisfies the heat equation and boundary conditions

$$v_t = v_{xx}$$

$$v(0) = v(1) = 0$$

And the initial condition is:

$$v(0,x) = \begin{cases} \frac{3x}{2} & 0 < x < \frac{2}{3} \\ 3 - 3x & 2/3 < x < 1 \end{cases}$$

the solution to this problem is:

$$u(t,x) = v(t,x) + x$$

the function $v(t,x)$ can be written as :

$$v(t,x) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

the coeff A_n is given by : $\quad A_n = 2\int_0^1 v(0,x)\sin(n\pi x)\,dx$

By the piecewise definition of $v(0,x)$, we compute :

$$a_n = 2\int_0^{2/3} \frac{3x}{2}\sin(n\pi x)\,dx = 3\int_0^{2/3} x\sin(n\pi x)\,dx$$

$$= 3x\,\frac{-\cos(n\pi x)}{n\pi}\Big|_0^{2/3} + 3\int_0^{2/3}\frac{\cos(n\pi x)}{n\pi}\,dx$$

$$= -2\,\frac{\cos\left(\frac{2n\pi}{3}\right)}{n\pi} + 3\,\frac{\sin(n\pi x)}{n^2\pi^2}\Big|_0^{2/3}$$

$$= -2\,\frac{\cos\left(\frac{2n\pi}{3}\right)}{n\pi} + 3\,\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\pi^2}$$

the second piece is :

$$b_n = 2\int_{2/3}^{1} (3-3x)\sin(n\pi x)\,dx$$

$$= 6\int_{2/3}^{1} (1-x)\sin(n\pi x)\,dx = -6(1-x)\frac{\cos(n\pi x)}{n\pi}\Big|_{2/3}^{1}$$

$$- 6\int_{2/3}^{1}\frac{\cos(n\pi x)}{n\pi}\,dx$$

$$= 2\,\frac{\cos\left(\frac{2n\pi}{3}\right)}{n\pi} + 6\,\frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\pi^2}$$

Finally $A_n = a_n + b_n$

$$= -2 \frac{\cos\left(\frac{2n\pi}{3}\right)}{n\pi} + 3 \frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\pi^2} + 2 \frac{\cos\left(\frac{2n\pi}{3}\right)}{n\pi}$$

$$+ 6 \frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\pi^2} = 9 \frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\pi^2}$$

$$\Rightarrow u(t,x) = 9 \sum_{1}^{\infty} \frac{\sin\left(\frac{2n\pi}{3}\right)}{n^2\pi^2} e^{-n^2\pi^2 t} \sin(n\pi x) + x$$

Significance of this problem:

1. Evaluate students' knowledge of heat equation
2. Evaluate students' knowledge of Fourier Series Solution
3. includes piecewise ~~non~~ initial condition which is more interesting and could be a bit more challenging than an ordinary IC.
4. includes non-zero BC

Source:

"Partial Differential Equations: An Introduction", Walter A. Strauss 2nd Edition.

$f(x) = x \quad -L < x < L$

$f(-L) = f(L) = 0$

Find the Fourier series for this function

f is an odd function thus its Fourier coefficients are

$a_n = 0 \quad , \quad n = 0, 1, 2$

$$b_n = \frac{2}{L}\int_0^L x \sin\frac{nx}{L}\,dx$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left(\sin\frac{n\pi x}{L} - \frac{n\pi x}{L}\cos\frac{n\pi x}{L}\right)\Big|_{x=0}^{x=L}$$

$$= \frac{2L}{n\pi}(-1)^{n+1} \qquad n = 1, 2, \ldots$$

Therefore the Fourier series for f is

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$

The periodic function f is discontinuous at the points $\pm L, \pm 3L, \ldots$

At these points the series converges to 0

This is a good problem because it talks about Fourier series and even and odd functions and requires calculations as well as visualization.

Elementary Differential Equations and Boundary Value Problems

Final Exam Question Submission

Q:

For the function : $f(x) = x/2$, $x \in [0,1]$

a) Derive the Fourier Sine Series.

b) Show that the series obtained on the interval $[0,1]$ could have also been derived from the full Fourier ~~Sine~~ Series on the interval $[-1,1]$

Hint : Recall,

$$\int_{-1}^{1} f_{odd}(x)\,dx = 0 \quad , \quad \int_{-1}^{1} f_{even}(x)\,dx = 2\int_{0}^{1} f_{even}(x)\,dx$$

A: a) The Fourier Sine Series is of the form:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

where,

$$A_n = 2\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow A_n = 2\int_0^1 \frac{x}{2}\sin(n\pi x)\,dx$$

$$= \int_0^1 x\sin(n\pi x)\,dx$$

Using Integration by Parts

$$\int_0^1 x\sin(n\pi x)dx = \frac{-x}{n\pi}\cos(n\pi x)\Big|_0^1 + \int_0^1 \frac{\cos(n\pi x)}{n\pi}\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi) + \cancel{\frac{1}{n\pi}}\ \frac{\sin(n\pi x)}{(n\pi)^2}\Big|_0^1$$

$$= \frac{(-1)^n}{n\pi}\ ,\ n \geq 1$$

b) The full Fourier Series for $f(x)$ is:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$$

from this the fourier Cosine Series ~~is~~ Coeff's are:

$$A_n = \int_{-1}^{1} \frac{x}{2}\cos(n\pi x)\,dx$$

$$= 0$$

$f(x)$ is odd ~~and~~ ~~so is~~ $\cos(x)$ is even, therefore integrand is odd and as such the result is zero.

The Fourier Sine Series ~~is~~ coeffs are:

$$B_n = \int_{-1}^{1} \frac{x}{2}\sin(n\pi x)\,dx$$

$$= 2\int_0^1 \frac{x}{2}\sin(n\pi x)\,dx$$

because $f(x)$ ~~is~~ and $\sin(x)$ are both odd functions, the integrand is even and the above integral is obtained. This is the same integral as was solved in part (a) and so it is possible to ~~extend~~ derive ~~any~~ the respective Fourier series from the full Fourier series for the function $f(x)$