PRACTICE PROBLEMS FOR MIDTERM

MATH 319 WINTER 2016

1. By using a method of your choice, solve the PDE
$$u_x \cos y + u_y = -u,$$
with the side condition $u(x,0) = e^{-x^2}$. Sketch the characteristic curves. Interpreting the y variable as time, describe in words how the initial "bump" function e^{-x^2} evolves as time runs.
2. Write down a solution of the heat equation
$$u_t = u_{xx},$$
for $0 \leq x \leq \pi$ and $t > 0$, satisfying the homogeneous Dirichlet boundary conditions
$$u(0,t) = u(\pi,t) = 0, \qquad (t > 0),$$
and the initial condition
$$u(x,0) = \sum_{n=1}^{N} \frac{1+(-1)^n}{n^s} \sin \frac{nx}{2}, \qquad (0 \leq x \leq \pi),$$
where N is a positive integer, and s is a real number, both considered to be given. *Hint*: The usual formula with product solutions would not directly apply, because $\frac{nx}{2}$ is not of the form mx with integer m.
3. Consider the PDE
$$u_t = \kappa u_{xx} + \alpha u + f,$$
on the spatial interval $0 < x < L$ (with, say $t > 0$), where κ, α, and L are positive constants, and f is a given function. By a change of variables, transform the problem into an equivalent problem
$$v_t = \varepsilon v_{xx} + v + g,$$
on the spatial interval $0 < x < 1$. Give formulas relating the new quantities v, ε, and g to the old ones.
4. Solve the wave equation
$$u_{tt} = u_{xx},$$
for $0 \leq x \leq \pi$ and $-\infty < t < \infty$, satisfying the boundary conditions
$$u(0,t) = u(\pi,t) = 0, \qquad (t > 0),$$
and the initial conditions
$$u(x,0) = 0, \qquad u_t(x,0) = \sum_{n=1}^{N} \frac{1+(-1)^n}{n^s} \sin \frac{nx}{2}, \qquad (0 \leq x \leq \pi),$$
where N is a positive integer, and s is a real number, both considered to be given.

5. Solve
$$u_{tt} = u_{xx},$$
for $0 \leq x < \infty$ and $-\infty < t < \infty$, satisfying the boundary condition
$$u(0,t) = 0, \qquad (-\infty < t < \infty),$$
and the initial conditions
$$u(x,0) = x^2, \qquad u_t(x,0) = 0, \qquad (0 \leq x < \infty).$$

6. Let $u(x,t)$ satisfy the heat equation for $x \in (0,1)$ and $t > 0$, the boundary conditions $u(0,t) = u_x(1,t) = 0$ for $t \geq 0$, and the initial condition $u(x,0) = f(x)$ for $x \in [0,1]$ with f a continuously differentiable function. Show that
$$\int_0^1 |u(x,t)|^2 \mathrm{d}x \leq \int_0^1 |f(x)|^2 \mathrm{d}x, \qquad \text{for any} \quad t \geq 0.$$
Derive a uniqueness theorem for the above initial-boundary value problem.