# SOLUTIONS TO SELECTED PRACTICE MIDTERM PROBLEMS

## Question 1

**Statement.** By using a method of your choice, solve the PDE

$$u_x \cos y + u_y = -u,$$

with the side condition $u(x,0) = e^{-x^2}$. Sketch the characteristic curves. Interpreting the $y$ variable as time, describe in words how the initial "bump" function $e^{-x^2}$ evolves as time runs.

**Solution.** If we want to describe any characteristic curve by a function $y = y(x)$, then we would have to solve $y'(x) = 1/\cos y$. One can of course proceed and solve the problem in this way, but one has to be extra careful because of the zeroes of $\cos y$. Let us choose a bit different approach, and describe the characteristic curves by $x = x(y)$. Then the characteristic equation is

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y,$$

which can easily be solved as

$$x(y) = \sin y + C = \sin y + x(0),$$

where we have identified the constant $C$ as the value of $x(y)$ at $y = 0$.

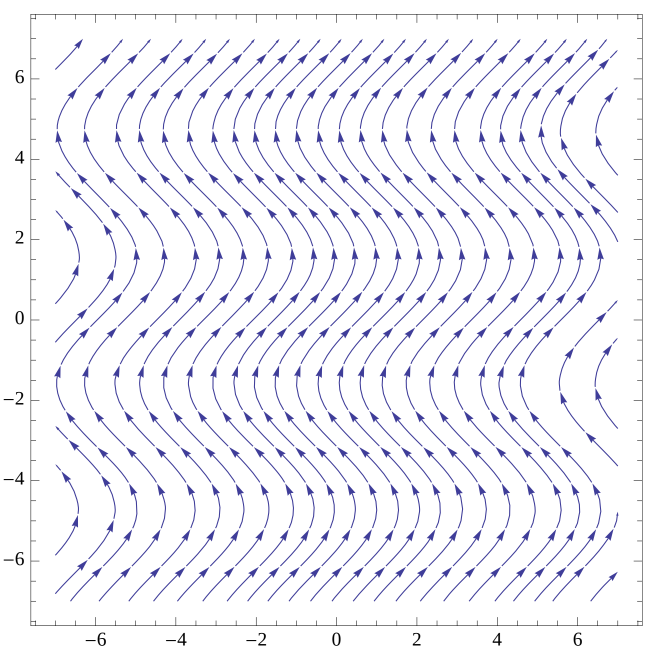


Figure 1. Characteristic curves for Question 1. The $x$-axis runs horizontally, and the $y$-axis vertically.

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*Date*: Math 319 Winter 2016.

Supposing that $u$ is a solution of the PDE, and by differentiating $u$ along a characteristic, we get

$$\frac{\mathrm{d}}{\mathrm{d}y}u(x(y),y) = u_x(x(y),y)\frac{\mathrm{d}x(y)}{\mathrm{d}y} + u_y(x(y),y) = -u(x(y),y),$$

and so

$$u(x(y),y) = Be^{-y}.$$

Plugging in the side condition gives

$$u(x(0),0) = B = e^{-x(0)^2}.$$

Now suppose that $(x,y)$ is an arbitrary point in $\mathbb{R}^2$. Then the characteristic curve passing through this point must have $x(0) = x - \sin y$. From this, we infer the solution

$$u(x,y) = e^{-(x-\sin y)^2}e^{-y}.$$

As time evolves, "bump" would decay (or "flatten") exponentially. At the same time, the peak of the "bump" would move back and forth in $x$-space by the law $x = \sin y$.

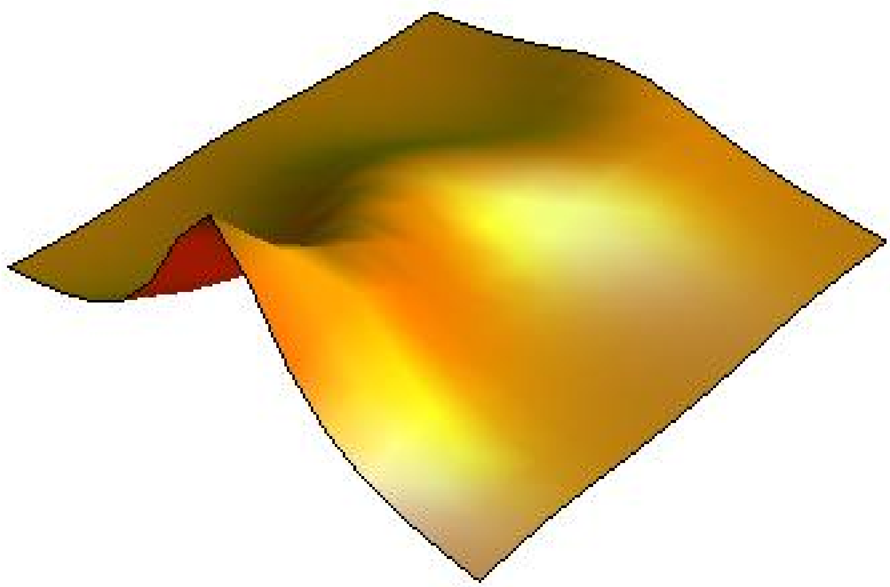

Figure 2. Depiction of the solution for Question 1. The $x$-axis is the one directed from top left to bottom right, and the $y$-axis is from bottom left to top right. The value of $u(x,y)$ is the elevation, which is a bit exaggerated for visualization purposes. In true scale $u(x,t)$ decays in time so fast that the wiggly movement of the peak would not be noticeable.

## Question 2

**Statement.** Write down a solution of the heat equation

$$u_t = u_{xx},$$

for $0 \leq x \leq \pi$ and $t > 0$, satisfying the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0, \qquad (t > 0),$$

and the initial condition

$$u(x,0) = \sum_{n=1}^{N} \frac{1+(-1)^n}{n^s} \sin\frac{nx}{2}, \qquad (0 \leq x \leq \pi),$$

where $N$ is a positive integer, and $s$ is a real number, both considered to be given. *Hint*: The usual formula with product solutions would not directly apply, because $\frac{nx}{2}$ is not of the form $mx$ with integer $m$.

**Solution.** The key observation here is that $1+(-1)^n = 0$ for odd $n$. This leaves only the even terms, hence the initial condition is

$$u(x,0) = \sum_{m=1}^{\lfloor N/2 \rfloor} \frac{2}{(2m)^s} \sin\frac{2mx}{2} = \sum_{m=1}^{\lfloor N/2 \rfloor} \frac{2}{(2m)^s} \sin mx, \qquad (0 \leq x \leq \pi),$$

where $\lfloor N/2 \rfloor$ is the largest integer not exceeding $N/2$. Now there is no problem with applying the product solution formula, and we get

$$u(x,t) = \sum_{m=1}^{\lfloor N/2 \rfloor} \frac{2e^{-m^2 t}}{(2m)^s} \sin mx, \qquad (0 \leq x \leq \pi,\ t \geq 0).$$

## Question 3

**Statement.** Consider the PDE

$$u_t = \kappa u_{xx} + \alpha u + f,$$

on the spatial interval $0 < x < L$ (with, say $t > 0$), where $\kappa$, $\alpha$, and $L$ are positive constants, and $f$ is a given function. By a change of variables, transform the problem into an equivalent problem

$$v_t = \varepsilon v_{xx} + v + g,$$

on the spatial interval $0 < x < 1$. Give formulas relating the new quantities $v$, $\varepsilon$, and $g$ to the old ones.

**Solution.** Let us put

$$v(x,t) = u(Lx, \lambda t),$$

where $\lambda$ is a real number that will be determined. Note that the scaling factor $L$ for the $x$ variable is immediately obvious, as we want $x$ in $v(x,t)$ to vary between $0 < x < 1$. We have

$$v_t(x,t) = \lambda u_t(Lx, \lambda t), \qquad v_{xx}(x,t) = L^2 u_{xx}(Lx, \lambda t),$$

and so

$$\begin{aligned} v_t(x,t) = \lambda u_t(Lx,\lambda t) &= \kappa\lambda u_{xx}(Lx,\lambda t) + \alpha\lambda u(Lx,\lambda t) + \lambda f(x,t) \\ &= \kappa\lambda L^{-2} v_{xx}(x,t) + \alpha\lambda v(x,t) + \lambda f(x,t). \end{aligned}$$

In order to have the coefficient in front of $v$ equal to 1, we need $\lambda = \alpha^{-1}$. This fixes the rest of the coefficients, and we conclude that the transformations

$$v(x,t) = u(Lx, \frac{t}{\alpha}), \qquad g(x,t) = \frac{f(x,t)}{\alpha}, \qquad \varepsilon = \frac{\kappa}{\alpha L^2},$$

satisfy the required conditions.